ELASTIC CONTACT PRESSURE UNDER FLAT PUNCHES HAVING SMOOTH CROSS-SECTIONS

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ABSTRACT
Surface contacts modeled by a flat ended rigid punch pressed on an elastic half-space possess analytical solution only for circular and elliptical cross-sections. This paper extends analytical solutions to the class of punch cross-sections bounded by mathematically smooth curves and applies the theory to Cassini’s ovals contours.

INTRODUCTION
Many technical applications contain conformal contacts modeled by a flat end, rigid punch pressed against an elastic half-space. Analytical solutions for this model exist only for circular and elliptical punch cross-sections [1]. These predict a minimum pressure in contact centre and pressure singularity in contour points. In other cases, Solomon [2] defines contact pressure as the ratio of a correction function to the square root of torsion Prandtl function of the cross-section. Mossakovski et al. [3] used for the nominator a function dependent on cross-section, expressed as a sum of three trigonometric series; the square root of difference between unity and the square of the ratio between polar contour radius and current polar radius, at the same polar angle, enters the denominator. Fabrikant [4] considers a constant nominator in Mossakovski et al. formulae. He argues that this is correct for circular and elliptical cross-sections but leads to errors for other domains.

By using Solomon result for polygonal contours, this paper proofs that Fabrikant method is accurate for the wider class of mathematically smooth contours. In this case, general formulae are derived for pressure distribution, central pressure and half-space penetration. These are applied for punches having Cassini ovals as cross-section contours.

DOMAINS BOUNDED BY SMOOTH CONTOURS
In the case the punch cross-section is a regular polygon, Solomon [2] uses following expression as pressure nominator:

\[ c(x, y) = A^k \left( \prod_{i=1}^{n} d_i(x, y) \right)^m, \]

where A is cross-section area, n is the number of polygon apexes, \( d_i(x, y) \) is the distance from a current point in contact area to the apex i, m is an exponent increasing with n, and k is a constant which assures that the right side member has units of length. Solomon recommends k=1/8 and m=1/4 for the triangle and k=1/10 and m=1/5 for the square. These values lead to following equations for k and m:

\[ m = (n + 1)^{-1}; \quad k = 0.5(n + 1)^{-1}. \]

If the number of sides, equal to n, tends to infinity, the polygon becomes a smooth closed contour and the function \( c(x,y) \) becomes a constant. It can be concluded that for punch cross-sections bounded by mathematically smooth contours (continuous and with continuous tangent) the pressure distribution advanced by Fabrikant is correct.

GENERAL FORMULAE
The contact pressure can be written now as follows:

\[ p(\rho, \theta) = p_0 \left( 1 - \frac{\rho^2}{\rho_c(\theta)} \right)^{-1/2}, \]

\( p_0 \) being central pressure and \( \rho \) and \( \rho_c \) correspond to the same polar angle. As Eq. (3) shows, \( p_0 \) is minimum contact pressure.

The integral entering force balance equation:

\[ p_0 \iint_A \left( 1 - \frac{\rho^2}{\rho_c(\theta)} \right)^{-1/2} \rho d\rho d\theta = Q \]

has following value:

\[ \iint_A \left( 1 - \frac{\rho^2}{\rho_c(\theta)} \right)^{-1/2} \rho d\rho d\theta = 2\pi \int_0^{\rho_c(\theta)} \rho d\rho = 2A. \]

Consequently, minimum contact pressure becomes half of average pressure:

\[ p_0 = \frac{Q}{2A}. \]

As half-space penetration is constant over entire contact area, namely \( \delta \), following equation can be written in the origin:
\[ \int_{0}^{2\pi} \left(1 - \frac{\rho^2}{\rho^2_{e}(\theta)}\right)^{-1/2} d\theta = \frac{\pi}{\eta p_0} \delta, \]  

(7)

\( \eta \) being the elastic constant of the contact, defined in terms of elastic parameters: 

\[ \eta = \left(1 - \nu^2\right)/E_e = \left(1 - \nu^2\right)/E_1 + \left(1 - \nu^2\right)/E_2. \]

The substitution \( \rho = \rho_e(\theta) \sin \alpha \) in the integral with respect to \( \rho \) entering Eq. (7) leads to the following relation:

\[ \int_{0}^{2\pi} \rho_e(\theta) d\theta = \frac{2 \delta}{\eta p_0}. \]

(8)

The integral in Eq. (8) has units of length and it is called the characteristic length of cross-section, being denoted by \( L^* \):

\[ L^* = \int_{0}^{2\pi} \rho_e(\theta) d\theta. \]

(9)

As a result, half-space penetration is given by:

\[ \delta = \frac{1}{2} \eta p_0 L^*. \]

(10)

In order to find all elements of an initially flat contact, one must use Eqs. (3), (6) and (10), prior to computing the characteristic length by means of Eq. (9).

APPLICATIONS

The application of the above method to elliptical and hence to circular cross-sections is straightforward. Pressure distribution yields from Eq. (3) in which the current point in contact area has following co-ordinates:

\[ x = ar \cos \alpha; \quad y = br \cos \alpha; \quad r \in [0, 1]; \quad \alpha \in [0, 2\pi], \]

(11)

\( a \) and \( b \) being ellipse semi-axes and \( r=1 \) corresponds to ellipse contour. Following known equations are found for pressure distribution, minimum pressure and characteristic length:

\[ p(\rho, \theta) = \frac{p_0}{\sqrt{1 - \rho^2}} \left[1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right]^{1/2}. \]

(12)

\[ p_0 = \frac{Q}{2\pi ab}, \]

(13)

\[ L^* = 4bK(e), \]

(14)

\( K \) being complete elliptic integral of first kind and \( e \) the eccentricity of the ellipse.

Other application of the theory concerns cross-sections bounded by Cassini ovals. These are smooth curves described in polar co-ordinates by following equation:

\[ \rho_e(\theta) = \sqrt{c^2 \cos(2\theta) + a^4 - c^4 \sin^2(2\theta)}. \]

(15)

The shape of Cassini ovals depends on parameters \( a \) and \( c \). If \( a \geq c\sqrt{2} \), a single convex oval occurs as in Fig. 1a, whereas for \( c < a < c\sqrt{2} \) a convex-concave shape occurs, as in Fig. 1b.

Following equations are found for pressure, area bounded by a Cassini oval, central pressure and characteristic length:

\[ \rho(\rho, \theta) = p_0 \left[1 - \frac{\rho^2}{c^2 \cos(2\theta) + a^4 - c^4 \sin^2(2\theta)}\right]^{1/2}; \]

(16)

\[ A = \frac{1}{2} \int_{0}^{2\pi} \rho_e^2(\theta) d\theta = 2a_1^2 E\left(\frac{c^2}{a^2}\right); \]

(17)

\[ p_0 = \frac{Q}{2A} = \frac{Q}{4a^2 E(c^2/a^2)}; \]

(18)

\[ L^* = \int_{0}^{2\pi} \sqrt{c^2 \cos(2\theta) + a^4 - c^4 \sin^2(2\theta)} d\theta, \]

(19)

\( E \) being the complete elliptic integral of second kind.

Two distributions of contact pressure are shown in Fig. 2, a and b, for \( c=1 \) and \( a=\sqrt{2} \) and \( a=1.05 \).

CONCLUSIONS

The accurate formula advanced by Fabrikant for elliptic and circular contacts is valid for the wider class of smooth contours. The pressure distribution, central pressure and half-space penetration in such situations can be calculated via simple formulae inhere obtained.

REFERENCES


