POISSON RATIO EFFECTS ON THE VON MISES AND MAXIMUM SHEAR STRESSES IN CYLINDRICAL CONTACTS

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ABSTRACT
This work determines the location of the greatest elastic distress in cylindrical contacts based upon the distortion energy and the maximum shear stress theories. The ratios between the maximum pressure, the von Mises stress, and the maximum shear stress are determined and fitted by empirical formulations for a wide range of Poisson ratios, which represent material compressibility. Some similarities exist between cylindrical and spherical contacts, where for many metallic materials the maximum von Mises or shear stresses emerge beneath the surface. However, if any of the bodies in contact is excessively compressible the maximum von Mises stress appears at the surface. That transitional Poisson ratio is found. The critical force per unit length that causes yielding onset, along with its corresponding interference and half-width contact are derived.

INTRODUCTION
Spherical contact has received considerable attention in the last four decades particularly because of its applicability to asperity contact. Using the meaning of hardness (Tabor [1]) a critical interference and plasticity index are derived by GW [2]. It is well known that for spherical contact the maximum von Mises stress occurs under the surface, and thus yielding onset would take place at that location. Similar to GW [2], CEB [3] present the critical interference that would cause such yielding

\[ \omega_s = \left( \frac{\pi KH}{2E} \right) R \left[ \frac{1}{1 - \nu_1^2} + \frac{1 - \nu_1^2}{E_1} + \frac{1}{R} \right] \]

(1 - 3)

where \( E_1, \nu_1, R_1, \nu_2, R_2 \) are the elastic properties and radii of sphere 1 and 2, respectively. This critical interference is identical to that given by GW [2] if one lets the hardness factor \( K=0.6 \). To accommodate various Poisson ratios a linear expression is proposed, \( K = 0.454 + 0.41\nu \), which is derived based upon broad assumptions regarding hardness. Without resorting to such assumptions an alternate derivation by JG [4] uses the von Mises yield criterion at the site of yielding, giving,

\[ \omega_s = \left( \frac{\pi C} {2E} \right) R \]

(4)

where \( C \) is found to be

\[ C = 1.295 \exp(0.736\nu) \]

(5)

and \( \nu \) is the Poisson ratio of the material that yields first (i.e., is either \( \nu_1 \) or \( \nu_2 \)). In contrast to Eq. (1), the critical interference in Eq. (4) relies directly upon the well-defined yield strength material property, \( \sigma_y \). The true interpretation of \( C \) is that it represents the ratio between the maximum contact pressure and the maximum von Mises stress, \( C = p_o / \sigma_{y,max} \). At yielding \( p_o \) takes on the critical value \( p_{oc} \), and by definition \( \sigma_{y,max} = S_y \), giving \( C = p_{oc} / S_y \). The maximum von Mises stress occurs under the surface at some location, \( z \), which varies with \( p_{oc} \). However, the value of \( C \) is determined for the ratio of \( dz/a \), (\( a \) being the contact radius) and thus is independent of \( z \) by itself.

The modeling of various machine elements such as gears, tapered rolling element bearings, or wheel-on-rail, lends itself to a cylindrical form of Hertzian contact. An equivalent determination of the expression for \( C \) is sought in this work for a cylindrical contact. It is found that maximum distress may occur at the surface or beneath it depending on Poisson’s ratio.

CYLINDRICAL HERTZIAN STRESS
Let \( x \) be the axis along the line of contact, \( y \) axis is tangent to the two cylinders, and \( z \) axis is the coordinate into the cylinders. The maximum (and principal) stresses occur at \( x=y=0 \). Under the total load, \( P \), maximum pressure is generated at the origin (Johnson [5])

\[ p_o = \frac{2P}{\pi bL} \]

(6)

where the Hertzian half-width is given by:

\[ b = \left( \frac{4(\lambda_1 + \lambda_2)PR_1 R_2}{L(R_1 + R_2)} \right)^{1/2} \frac{4PR}{\pi LE} \]

\[ \lambda_i = 1 - \nu_i^2 \]

(7)

Defining \( \zeta = \frac{z}{b} \), then the stresses for cylindrical contact are

\[ \sigma_y = - \left( -2\zeta + \sqrt{1 + \zeta^2} \left( 2 - \frac{1}{1 + \zeta^2} \right) \right) p_o \]

(8)

\[ \sigma_z = - \frac{p_o}{\sqrt{1 + \zeta^2}} \]

(9)

These stresses are calculated in either material 1 or 2, where clearly \( \zeta \geq 0 \) in both materials. These stresses are independent of Poisson’s ratio. Assuming the state of plain strain the transverse stress is \( \sigma_z = \nu(\sigma_y + \sigma_x) \) which reduces to,

\[ \sigma_z = -2 \left( -\zeta + \sqrt{1 + \zeta^2} \right) \sigma_x \]

(10)

While the theoretical limit of Poisson’s ratio is between \(-1 < \nu \leq 1/2 \), it is rare to encounter engineering materials with negative Poisson ratios. Most materials will fall in the range \( 0 \leq \nu \leq 1/2 \) and the discussion herein is limited to this range. In the above, letting \( \nu \) approach zero leads to a bi-axial stress state (e.g., plane stress).
MAXIMUM VON MISES STRESS AND CRITICAL PARAMETERS

Based upon these relationships, the von Mises stress, $\sigma$, is calculated by:

$$\sigma = \sqrt{\frac{1}{4} \left[ -1 + 2 \zeta \left( \sqrt{1 + 2 \zeta} - 1 \right) \right]^2 + \left( 1 + 4 \zeta^2 + 4 \right) \left( -1 + \nu \right) \frac{1}{1 + \zeta^2}}$$  \hspace{1cm} (11)

The above varies with $\zeta$ where $\nu$ is a parameter. A plot of this ratio is given in Fig. 1.

![Fig. 1: Normalized von Mises stress as a function of nondimensional $|z/b|$ for various Poisson ratios.](image)

It is noted that the maximum von Mises stress can occur either on the surface ($z/b=0$) or somewhere under the surface ($z/b>0$) depending upon Poisson’s ratio. The threshold is found below, but the fact that in cylindrical contact the maximum von Mises stress may be positioned at the surface is unique for cylindrical contact (opposed to spherical contacts where the maximum von Mises stress is always under the surface regardless of Poisson ratio). The reason for this happening in the case of cylindrical contact is imbedded in the assumption of plane strain where $\sigma_z$ is linearly dependent upon the Poisson ratio (see Eq. (10)). As $\nu$ diminishes the tri-axial stress state tends to a bi-axial stress state, leading to a higher von Mises stress for the same load, $P$ or $p_w$. At $\zeta = 0$ Eq. (11) degenerates to

$$\frac{\sigma}{p_{c}} = \sqrt{1 + 4(\nu - 1)\nu}$$ \hspace{1cm} (12)

For perfect compressibility $\nu = 0$, at $\zeta = 0$, a theoretical value is obtained, $\sigma / p_c = 1$.

The maximum von Mises stress is obtained by letting $d(\sigma / p_c) / d\zeta = 0$. This equations is solved numerically for $\zeta$ where $\nu$ as a parameter is varied in the aforementioned range. The value of $\zeta$ is substituted back into Eq. (11) for finding the maximum value of $\sigma / p_c = C^{-1}$, where clearly $C = C(\nu)$. As discussed above a global stationary point of maximum exists only above certain $\nu$. Up to that value the maximum is at the surface according to Eq. (12). A search reveals that at $\nu = 0.1938$, the transition occurs at a value of $C^{-1} = \sigma / p_c = 0.6122$ with an error of $10^{-4}$. The behavior of $\sigma / p_c$ for this value of $\nu = 0.1938$ is shown Fig. 1. Beyond $\nu = 0.1938$ the numerical values of $C(\nu)$ are curve fitted to a parabolic expression given below in Eq. (13), with almost indistinguishable error. In summary,

$$C = \begin{cases} \frac{1}{4 \nu^2 (1-\nu^2)^{1/2}} & : \zeta = 0 \\ \frac{1.164 + 2.975 \nu - 2.906 \nu^2}{\nu} & : \nu < 0.1938 \\ \frac{0.223 + 3.231 \nu - 2.397 \nu^2}{\nu} & : \nu > 0.1938 \end{cases}$$ \hspace{1cm} (13)

This value of $C$ is valid for as long as the material is elastic, i.e., up to yielding onset. This value is used to calculate critical parameters. Using the distortion energy (von Mises) theory to predict yielding onset, and letting $CS_{yo} = \min(C(\nu)S_{11}, C(\nu)S_{22})$, then the critical values for force per unit length, interference (using Hamrock [6]), and half-width are derived,

$$p_c = \frac{\pi R (CS_{yo})}{E} : \delta_c = R \left( \frac{CS_{yo}}{E} \right) \left[ 2 \ln \left( \frac{2E}{CS_{yo}} \right) - 1 \right] : b_c = \frac{2RCS_{yo}}{E}.$$ \hspace{1cm} (14)

MAXIMUM SHEAR STRESS

Oftentimes the maximum shear stress is used to determine distress. Defined as $\tau_{max} = \max|\sigma_y - \sigma_z| / \sigma_y$, then substitution of Eqs. (8-10) results in

$$\tau_{max} / p_c = \max \left| \frac{\zeta - \zeta' + (-2\zeta^2 + 4\nu \sqrt{1 + \zeta^2})/\sqrt{1 + \zeta^2}}{2 \sqrt{1 + \zeta^2}} \right|$$ \hspace{1cm} (15)

Fig. 2 indicates that the maximum shear stress occurs at the surface when $\nu = 0$. For $\nu > 0$ maximum value appears beneath the surface having a reduced maximum value as $\nu$ increases. The function of $\tau_{max}$ may contain a discontinuity in slope because of the nonlinearity in the $\max()$ function, leading to two stationary points. Again, the maximum shear stress depends upon the Poisson ratio.

![Fig. 2: Normalized maximum shear stress as a function of nondimensional $|z/b|$ for various Poisson ratios.](image)

REFERENCES