REYNOLDS EQUATIONS FOR COMMON GENERALIZED NEWTONIAN MODELS 
AND AN APPROXIMATE REYNOLDS-CARREAU EQUATION

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ABSTRACT
Exact, closed form one-dimensional Reynolds equations are presented for the Ostwald-DeWaele model, Ellis model, Spriggs model and the double-Newtonian Rabinowitsch and Ferry models. From numerical solutions for flow rate, an approximate Reynolds-Carreau equation is obtained.

1. INTRODUCTION
The Newtonian film thickness formulas for concentrated contact lubrication have been shown to be accurate for low-molecular-weight liquids only. When a polymer is blended with the base oil or when the base oil itself is of high molecular weight, these formulas overestimate film thickness. The reason for the discrepancy was shown, forty years ago\(^1\), to be shear-thinning of the lubricant. Rheometers capable of characterizing shear-thinning under pressure have been available for nearly the same period of time\(^2\). Measurements have demonstrated that shear-thinning response follows, to the best approximation, a power-law which is sometimes followed by a second Newtonian plateau. Of course, for an accurate calculation of traction, the same shear-thinning model is required. Progress toward obtaining Reynolds equations for common shear-thinning models has been slow. For EHL film thickness calculations, often a direct integration across the film is utilized instead\(^3\).

This paper presents line contact Reynolds equations for some generalized Newtonian models that have stress as the independent variable. Expressions for shear rate and velocity are integrated across the film to yield an equation for flow rate in sliding contact. Exact, closed form expressions are presented for the Ostwald-DeWaele model, Ellis model, Spriggs model and the double-Newtonian Rabinowitsch and Ferry models. Finally, from numerical solutions for flow rate, an approximate Reynolds-Carreau equation is obtained.

2. APPROACH AND NOMENCLATURE
The problem is non-dimensionalized as follows: The shear stress, \(\tau\), becomes \(\dot{\tau} = \frac{\tau}{G}\) and the shear rate becomes \(\dot{\gamma} = \frac{\mu}{G} \) where \(\mu\) is the limiting low shear viscosity. Reduced velocities are given by \(\hat{u}_1 = \frac{u_1}{hG}\) where \(h\) is the local film thickness. The slide-to-roll ratio is \(\bar{\Sigma} = (u_2 - u_1)/\bar{u}\) where \(\bar{u} = (u_2 + u_1)/2\). The dimensionless generalized viscosity becomes \(\hat{\eta} = \eta / \mu\) and the second Newtonian viscosity, \(\mu_2\), becomes \(\hat{\mu} = \frac{\mu_2}{\mu}\). The pressure gradient in the flow direction, \(p'\), becomes \(\hat{p}' = \frac{p'h}{G}\) and the volume flow rate is \(\hat{Q} = \frac{Qh^2}{G}\). The shear stresses at surfaces 1 and 2 are \(\tau_1 = \tau_m - \frac{p'}{2}\) and \(\tau_2 = \tau_m + \frac{p'}{2}\). In dimensional terms, the flow rate can be written \(Q = \bar{m}h\), where \(h\) is the film thickness where \(\bar{p} = 0\) and this leads to the substitution \(\bar{u} - \dot{\bar{Q}}(h-h_0)/h\) which transforms an expression for \(\dot{Q}\) into the integrated form of a one-dimensional Reynolds equation.

3. RESULTS
Here the model will be presented along with any restrictions. Then, if available, an expression for the dimensionless mid-plane stress, \(\tau_m\) will be given. Finally, the expression for flow rate will be presented.

3.1. Models with no transition
3.1.1 Newtonian
\[ \dot{\eta} = 1, \quad \dot{\tau}_m = \sum \dot{\tau} \quad \dot{Q} = \dot{\tau} - \frac{\dot{\tau}^2}{12} \text{ or } \frac{2\dot{\tau}^2}{12} = \sum \dot{\tau} \left( \frac{h - h_c}{h} \right) \]

3.1.2 Ostwald-DeWaele

\[ \dot{\eta} = \left| \dot{\tau} \right| \frac{1}{n}, \quad \dot{\tau}_m = \sum \dot{\tau} - \frac{\dot{\tau}^{1 + \frac{1}{n}}}{1 + \frac{1}{n}} \]

\[ \dot{Q} = \dot{\tau} - \frac{\dot{\tau}^2}{12} \left[ \left( \frac{\dot{\tau}}{1 + \frac{1}{n}} \right)^{\frac{1}{n}} - \left( \frac{\dot{\tau}}{1 + \frac{1}{n}} \right) \right] \]

3.2. Shear thinning models with a first Newtonian

3.2.1 Spriggs, \( \Sigma = 0 \)

\[ \dot{\eta} = \begin{cases} 1, & \left| \dot{\tau} \right| < 1 \\ \left| \dot{\tau} \right| \frac{1}{n}, & \left| \dot{\tau} \right| > 1 \end{cases} \]

\[ \dot{Q} = \dot{\tau} - \frac{\dot{\tau}^2}{12} \left[ \frac{\dot{\tau}}{1 + \frac{1}{n}} \right]^{\frac{1}{n}} - \left[ \frac{\dot{\tau}}{1 + \frac{1}{n}} \right] \]

3.2.2 Ellis

\[ \dot{\eta} = \frac{1}{1 + \left| \dot{\tau} \right|}, \quad \dot{\tau}_m = \sum \dot{\tau} - \frac{\dot{\tau}^{1 + \frac{1}{n}} - \left( \frac{\dot{\tau}}{1 + \frac{1}{n}} \right)^{1 + \frac{1}{n}}}{\dot{\tau}^{1 + \frac{1}{n}}} \]

\[ \dot{Q} = \dot{\tau} - \frac{\dot{\tau}^2}{12} \left( \frac{\dot{\tau}}{1 + \frac{1}{n}} \right)^{\frac{1}{n}} - \left( \frac{\dot{\tau}}{1 + \frac{1}{n}} \right) \]

3.2.3 Carreau, \( \left| \dot{\tau} \right| < 30 \)

\[ \dot{\eta} = [1 + \dot{\tau}^2]^{\frac{n-1}{2}}, \quad \dot{\tau}_m = \sum \dot{\tau} \left( 1 + 0.053 \dot{\tau}^4 \right)^{\frac{1}{n-1}} \left[ 1 + \left( \sum \dot{\tau} \right) \right]^{\frac{n-1}{2}} \]

\[ \dot{Q} = \dot{\tau} - \frac{\dot{\tau}^2}{12} \left( \frac{\dot{\tau}}{1 + \frac{1}{n}} \right)^{\frac{1}{n}} \]

3.3. Shear thinning models with two Newtonians

3.3.1 Double Newtonian Rabinowitsch model

\[ \dot{\eta} = \dot{\mu} + \frac{1 - \dot{\mu}}{1 + \dot{\tau}^2}, \quad \dot{\tau}_m = \dot{\mu} \sum \dot{\tau} + \frac{1}{2 \mu} \left( \dot{\mu} + \tau_2 \right) \left( \dot{\mu} + \tau_1 \right) \]

\[ \dot{Q} = \dot{\tau} - \frac{\dot{\tau}^2}{12} \left( \frac{\dot{\tau}}{1 + \frac{1}{n}} \right)^{\frac{1}{n}} - \left( \frac{\dot{\tau}}{1 + \frac{1}{n}} \right) \]

3.3.2 Double Newtonian Ferry model

\[ \dot{\eta} = \mu + \frac{1 - \mu}{1 + \dot{\tau}^2}, \quad \dot{\tau}_m = \mu \sum \dot{\tau} + \frac{1}{\mu} \left( \frac{1 - \mu}{\mu} \right) \left( \frac{1}{\mu} \right) \left( \dot{\mu} + \tau_2 \right) \left( \dot{\mu} + \tau_1 \right) \]

\[ \dot{Q} = \dot{\tau} - \frac{\dot{\tau}^2}{12} \left( \frac{\dot{\tau}}{1 + \frac{1}{n}} \right)^{\frac{1}{n}} - \left( \frac{\dot{\tau}}{1 + \frac{1}{n}} \right) \]

4. CONCLUSIONS

All of the above equations can be solved numerically for the pressure gradient in terms of the average velocity, film thickness and in some cases the sliding velocity. These equations have been employed in a simple Grubin inlet zone calculation for central film thickness. Film thickness was also calculated by direct integration of the Carreau equation for comparison. The approximate Reynolds-Carreau form accurately determines film thickness, however, the Reynolds-Spriggs form is also an accurate approximation to the Carreau model. On the other hand, the Reynolds-Ellis equation is only an accurate approximation for a Carreau fluid when \( n \) is small. These relationships should prove useful for the calculation of film thickness when the lubricant shears within the inlet zone.

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REFERENCES