ON THE KINEMATICS AND KINETICS OF MECHANICAL SEALS, ROTORS, AND WOBBLING BODIES

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ABSTRACT
Mechanical seals, rotors, and wobbling bodies are characterized by a kinematical constraint that prevents them from having integral motion with respect to their own frame. A valid kinematical model is a prerequisite to subsequent dynamic analyses. Three previous works have suggested distinctly different kinematical models to the same problem. The analysis herein presents yet another kinematical model that preserves (actually enforces) the proper kinematical constraint. The outcome reaffirms one of the previous models. The equations of motion are derived using Lagrange’s equations to complement results obtained previously by Newton-Euler mechanics.

INTRODUCTION
Mechanical seals have typically at least one element flexibly mounted; it is either the stator or the rotor [1]. Rotors (flywheels, impellers, etc.) are typically attached to flexible rotating shafts. These are supported by some mechanism that allows them to whirl about a point, but it does not allow them to complete an integral revolution about that point. For example, in mechanical seals there are anti-rotation pins that allow the flexibly mounted element track a misaligned element, while disks or rotors are fixed to a shaft, by various means (e.g., keyways, press fits, bolts, welding). Had there been an integral (complete) rotation, regardless of the application, the anti-rotation device (pins, keys, etc.) would shear off, a condition which is not permitted in a functional system. Elements that obey such constraints are classified as wobbling bodies.

Generally the “point” about which tilt takes place and motion is transmitted, in a basic sense, represents a joint (e.g., Cardan suspension, Hooke joint, or a constant velocity joints). However, the locking devices mentioned above are only a few out of the numerous mechanisms that fulfill the same or similar function. Instead of pins there can bellows, or O-rings that support the flexibly mounted element in a seal. Likewise, welding, bolts, or press-fits are also used to attach rotors to rotating shafts. Since anti-rotation pins allow free rotation (tilt motion) only in a prescribed order they are not axisymmetric joints. On the other hand bellows, O-rings, welds, and press fits do not impose any preferential order on rotations and can be considered axisymmetric (isotropic) joint. Clearly it is impossible to account for all mechanisms in a single kinematical model without compromise. If, however, one accepts the fact that tilts are limited to small angles, a single kinematical model is possible. There is one constraint, though, that cannot be violated. This is the fact that as long as the locking devices are functional, no shearing can take place between the wobbling body and the housing or shaft upon which the body is attached. This leads to a transmission law that must be satisfied. Note that this work is limited to rigid bodies, i.e., flutter that may be caused by elastic modes is not considered.

KINEMATICAL MODELS
Figure 1 presents a kinematical model for a flexibly mounted element that is constrained by two anti-rotation pins. Its angular position is described by the Euler angles, i.e., precession, nutation, and spin. Moments are sought in a rotating coordinate system, xyz, which precesses relative to coordinate system XYZ that is attached to the outer rim (or housing).

System XYZ is either fixed in space (flexibly mounted stator, FMS) or is rotating (flexibly mounted rotor, FMR) with the shaft having a velocity $\omega$ about $Z$. It is then further assumed that $\omega = \text{const}$ (for FMR) or $\omega = 0$ (for FMS). System $xyz$ is whirling (wobbling) within $XYZ$ such that $x$ is the axis about which the nutation (tilt) occurs, $z$ is the out-normal axis about which the spin takes place, and $y$ points toward the point of maximum separation between planes containing $XY$ and $xy$. 

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Fig.1: A kinematical model for a seal wobbling element

(a) 3D view
(b) 2D cross-section
Three different works have proposed kinematical constraints for mechanical seals that are fundamentally different. Reference [2] proposes a transmission law,

\[ T_1 = -\partial_t \psi \quad (1) \]

where Ref. [3] presents,

\[ T_2 = \cos \gamma \quad (2) \]

Earlier kinematical models with transmission laws are given in [1] for a constant velocity (isotropic) joint,

\[ T_3 = -\partial_t \psi = 1 \quad (3a) \]

and a Hooke joint (non-isotropic) operating under small tilts

\[ T_4 = -\partial_t \psi = 1 + (\sin^2 \psi - \frac{1}{2}) \gamma^2 - \frac{\gamma^2}{2} \sin \psi \cos \psi \quad (3b) \]

Note that since the kinematical constraint under consideration is between XYZ and xyz, it renders the issue of whether XYZ is inertial or rotating as immaterial. Another mechanism which guarantees that a point in the wobbling body returns to its original position after completing one cycle of whirl (abiding by the locking devices, gear teeth, etc.) the joint imposes

\[ \frac{1}{2\pi} \int_0^{2\pi} T \, d\psi = 1 \quad (4) \]

Only the two transmission laws in Eqs. (3) fulfill this constraint while those in Eqs. (1, 2) do not, and thus they are not applicable to wobbling bodies under the said constraint.

**LAGRANGES EQUATIONS**

The work by Green and Etsion [1] provides dynamic moments about xyz using Newton-Euler mechanics. Reference [4] provides the same equations using Lagrange’s equations; however, the outcome is enigmatic since the aforementioned kinematical constraint is unaccounted for. A classical way to include constraints of that nature would be through Lagrange multipliers. Here, however, the Lagrange equations will contain the kinematical constraint via Eqs. (20, 21) in Ref. [1],

\[ \dot{\psi} = \psi + \omega; \quad \dot{\phi} = \omega - \dot{\psi}, \]

both of which are imbedded in Eq. (22) in the said reference, giving eventually the angular velocity of the wobbling body,

\[ \ddot{\lambda} = \dot{\gamma}^2 + \dot{\psi}^2 + \sin \gamma \dot{\gamma} \dot{\psi} + [\psi, (\cos \gamma - 1) + \omega]^2 \quad (5) \]

The abc system (Fig. 1) is principal for the wobbling body possessing corresponding inertia values \( I_1, I_2, I_3 \), where \( I_1 \) and \( I_2 \) are the transverse and polar moment of inertia, respectively. The angular momentum is then \( \vv{I} = [I(\dot{X} + \dot{Y})] \), leading to the kinetic energy (where origin O, as center of mass, is stationary),

\[ T = \frac{1}{2} \int h \cdot \ddot{\lambda} = \frac{1}{2} \{ I \dot{\gamma}^2 + 2I \dot{\gamma} \dot{\psi}^2 + I \psi^2 \dot{\gamma} \psi^2 \} \quad (6) \]

The Lagrange equations are

\[ \frac{dT}{dq} = \frac{\partial L}{\partial q} \quad (7) \]

where \( \{q, q, q, q\} \equiv \{\gamma, \psi, \theta\} \) and \( \theta = \omega \). Obtaining the moment, \( M_n \), in the tilt direction is straightforward since the generalized coordinate, \( \gamma \), coincides with that direction. Since \( M_n \) is not along the generalized coordinates, it requires special treatment (which has not been worked out in the previous work). For this stationary problem of steady precession where the time-invariant constraint does not do work the generalized forces are obtained by (see Ginsberg [5], pp. 257-269),

\[ \dot{Q}_j = \sum \frac{\partial F}{\partial q_j} = \sum \frac{\partial \ddot{r}}{\partial q_j} = \sum M \frac{\partial \lambda}{\partial q_j} \quad (8) \]

with the second equality being inferred. Substituting Eq. (5) in (8), and letting \( j = 1, 2, 3 \), leads to

\[ \dot{Q}_j = M \quad (9) \]

Applying Eq. (7) to (6) and with the aid of (9) gives the equations of motion expressed in system abc for finite angles, \( I \dot{\gamma}^2 + I \dot{\psi}^2 \sin \gamma \cos \gamma + I \{ \psi \dot{\psi} \cos \gamma - 1 \} + \omega \dot{\psi} \sin \gamma \)

\[ I \dot{\psi} \sin \gamma + 2I \dot{\psi} \dot{\gamma} \psi \sin \gamma \cos \gamma + I \{ \dot{\psi} \cos \gamma - 1 \} \dot{\psi} \sin \gamma \]

\[ + \omega \dot{\psi} \{ \cos \gamma - 1 \} - \dot{\psi} \{ \cos \gamma - 1 \} + \omega \dot{\psi} \} \sin \gamma \]

\[ = M \sin \gamma + M \cos \gamma \]

Now these equations are simplified for small angles, where \( \sin \gamma = \gamma + O(\gamma^3) \), \( \cos \gamma = 1 - \gamma^2/2 + O(\gamma^4) \), yields ultimately in,

\[ M_1 = I (\dot{\gamma}^2 - \dot{\psi}^2 \gamma) + I \omega \dot{\psi} \gamma + O(\gamma^2) \quad (11) \]

\[ M_2 = I (\dot{\psi} + 2\dot{\phi} \gamma - \dot{\psi} \omega + O(\gamma^2) \]

\[ M_3 = -I (\dot{\psi} \dot{\gamma} + \dot{\psi} \gamma + \dot{\psi} \gamma + O(\gamma^2) = O(\gamma^2) \]

These equations match identically those in Eq. (24) in Ref. [1], which had been obtained by Newton-Euler mechanics.

**CONCLUSIONS**

A valid kinematical model that represents the physical constraint imposed by the anti-rotation or locking devices is fundamental to the proper derivation of the equations of motion. A new model is proposed. With the constraint imbedded in the angular velocity of the wobbling body the use of Lagrange’s equations affirms a previous result.

**REFERENCES**