DFT fiddle factor derivation

The Discrete Fourier Transform is given by

$$X_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x_n \exp\left(-\frac{2\pi i}{N} kn\right) \qquad k = 0, \ldots, N-1 \tag{1}$$

Or, given a vector, $\boldsymbol{x} = [x_0, x_1, \ldots x_{n-1}]^T$, its DFT, $\mathbf{X} = [X_0, X_1, \ldots, X_{k-1}]^T$ can be found using the DFT matrix, $\tilde{\boldsymbol{W}}$.

$$\boldsymbol{X} = \tilde{\boldsymbol{W}}\boldsymbol{x} \tag{2}$$

Where:

$$\tilde{\boldsymbol{W}} = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & w & \ldots & w^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & w^{N-1} & \ldots & w^{(N-1)(N-1)} \end{pmatrix} \qquad \text{and} \qquad w = \exp\left(-\frac{2\pi i}{N}\right) \tag{3}$$

If values x_n (where $n = 0, \ldots, N-1$) are drawn independently from a zero-mean Gaussian with variance σ^2 the mean of the resulting frequency-domain vector is given by:

$$\boldsymbol{\mu} = E[\boldsymbol{X}] = E[\tilde{\boldsymbol{W}}\boldsymbol{x}] = \tilde{\boldsymbol{W}}E[\boldsymbol{x}] = [0 \ \ldots \ 0]^T \tag{4}$$

Where $\boldsymbol{A}^H$ denotes the Hermitian (complex) transpose of $\boldsymbol{A}$, and $\boldsymbol{I}$ is the identity matrix, the covariance matrix of the frequency-domain vector can be found with:

$$\begin{aligned} \boldsymbol{\Sigma} &= E[(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^H)] \\ &= E[\boldsymbol{X}\boldsymbol{X}^H] \\ &= E[(\tilde{\boldsymbol{W}}\boldsymbol{x})(\tilde{\boldsymbol{W}}x)^H)] \\ &= E[\tilde{\boldsymbol{W}}\boldsymbol{x}\boldsymbol{x}^T\tilde{\boldsymbol{W}}^H] \\ &= \tilde{\boldsymbol{W}}E[\boldsymbol{x}\boldsymbol{x}^T]\tilde{\boldsymbol{W}}^H \\ &= \tilde{\boldsymbol{W}}\sigma^2\boldsymbol{I}\tilde{\boldsymbol{W}}^H \\ &= \sigma^2\tilde{\boldsymbol{W}}\tilde{\boldsymbol{W}}^H \\ &= \sigma^2\boldsymbol{I} \end{aligned} \tag{5}$$

We now introduce the refinement of a time-domain window, $\boldsymbol{f}$ (a column vector), such that now $\boldsymbol{x} = \boldsymbol{f} \circ \boldsymbol{y}$ and $\boldsymbol{y}$ is the observed (zero-mean Gaussian-distributed) data. In this context, "$\circ$" denotes the Hadamard (element-wise) product.

Alternatively we can say that $\boldsymbol{x} = \tilde{\boldsymbol{F}}\boldsymbol{y}$ where $\tilde{\boldsymbol{F}} = \boldsymbol{f}^D$, i.e. the window vector presented as a diagonal matrix.

Now:

$$\begin{aligned}
\boldsymbol{\Sigma} &= E[(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^H)] \\
&= E[\boldsymbol{X}\boldsymbol{X}^H] \\
&= E[(\tilde{\boldsymbol{W}}\tilde{\boldsymbol{F}}\boldsymbol{x})(\tilde{\boldsymbol{W}}\tilde{\boldsymbol{F}}x)^H)] \\
&= E[\tilde{\boldsymbol{W}}\tilde{\boldsymbol{F}}\boldsymbol{x}\boldsymbol{x}^T\tilde{\boldsymbol{F}}^H\tilde{\boldsymbol{W}}^H] \\
&= \tilde{\boldsymbol{W}}\tilde{\boldsymbol{F}}E[\boldsymbol{x}\boldsymbol{x}^T]\tilde{\boldsymbol{F}}^H\tilde{\boldsymbol{W}}^H \\
&= \tilde{\boldsymbol{W}}\tilde{\boldsymbol{F}}\sigma^2\boldsymbol{I}\tilde{\boldsymbol{F}}^H\tilde{\boldsymbol{W}}^H \\
&= \sigma^2\tilde{\boldsymbol{W}}\tilde{\boldsymbol{F}}\tilde{\boldsymbol{F}}^H\tilde{\boldsymbol{W}}^H \\
&= \sigma^2(\tilde{\boldsymbol{W}}\tilde{\boldsymbol{F}})(\tilde{\boldsymbol{W}}\tilde{\boldsymbol{F}})^H \\
&= \sigma^2\tilde{\boldsymbol{G}}\tilde{\boldsymbol{G}}^H \qquad \text{where } \tilde{\boldsymbol{G}} \text{ is the DFT of } \tilde{\boldsymbol{F}}
\end{aligned} \tag{6}$$

The diagonal nature of $\tilde{\boldsymbol{F}}$ produces a conjugate-symmetric $\boldsymbol{\Sigma}$ with terms not varying diagonally, that is to say $\boldsymbol{\Sigma}_{i,j} = \boldsymbol{\Sigma}_{i+1,j+1}\ \forall i,j \in [0,\dots N-2]$.

In the case of all useful window functions, $\boldsymbol{\Sigma}$ is dominated by its leading diagonal. In the case of the commonly used Hann window[1], 51.40% of the energy of $\boldsymbol{\Sigma}$ lies on the leading diagonal with 97.11% and 99.97% of the energy encompassed in the central three and five diagonals respectively (independent of window length).

For the sake of simplicity we neglect the covariance of adjacent spectral bins, instead considering only the variances of individual bins, found on the leading diagonal, i.e. $\hat{\boldsymbol{\Sigma}} = k\sigma^2\boldsymbol{I}$. We then find a suitable value for k to effectively convert the time-domain signal variance to a frequency-domain equivalent, subject to some error matrix, $\tilde{\boldsymbol{\epsilon}} = \boldsymbol{\Sigma} - \hat{\boldsymbol{\Sigma}}$:

$$\begin{aligned}
\boldsymbol{\Sigma} &= \hat{\boldsymbol{\Sigma}} + \tilde{\boldsymbol{\epsilon}} \\
\boldsymbol{\Sigma} &= k\sigma^2\boldsymbol{I} + \tilde{\boldsymbol{\epsilon}} \\
k &= \frac{1}{\sigma^2}\left(\boldsymbol{\Sigma} - \tilde{\boldsymbol{\epsilon}}\right)_{1,1}
\end{aligned}$$

A single value (1,1) from the leading diagonal of the $(\boldsymbol{\Sigma} - \tilde{\boldsymbol{\epsilon}})$ matrix is used, given the nature of $\boldsymbol{\Sigma}$ as described above. Neglecting the error term, we arrive at an approximation for k:

$$k \simeq \left(\tilde{\boldsymbol{G}}\tilde{\boldsymbol{G}}^H\right)_{1,1} \tag{7}$$

For a Hann window, this evaluates to approximately 0.3748. In other words, for time-domain and spectral bin variances σ_t^2 and σ_f^2 respectively:

$$\sigma_f^2 = 0.3748\sigma_t^2 \tag{8}$$

or

$$\sigma_t^2 = 2.6680\sigma_f^2 \tag{9}$$

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[1] where the nth element of $\boldsymbol{f}$ is given by $f_n = 0.5\left(1 - \cos\left(2\pi\frac{n}{N-1}\right)\right)$, $n \in [0,\dots N-1]$