A Survival Guide to Vector Calculus

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When I first tried to learn about Vector Calculus, I found it a nightmare. Eventually things became clearer and I discovered that, once I had really understood the 'simple' bits of the subject, the rest became relatively easy. This is my attempt to explain those 'simple' concepts clearly, in the hope that the same will happen for you!

1 Scalar Fields and Grad

A Scalar Field is the name given to any function in which a scalar quantity (e.g. temperature) is expressed as a function of spatial position (e.g. position in a room at a given instant in time). The 'space' in which a scalar field exists can have any number of dimensions: if it is one-dimensional then the function is just a simple expression such as $T = x^2 - 3x + 2$. A scalar field in a two-dimensional space (e.g. $T = x^2 - 3xy + 2$) can be drawn as a simple 'contour map', in which lines of constant T are plotted in the XY space. It can also be thought of as a kind of 'landscape', if T is plotted along the Z axis in a Cartesian reference frame. This can be useful, because it means we can describe the behaviour of the function using topographical terms such as peaks, valleys, and saddle points; but it can also be limiting, because T has been given the same dimensionality (i.e. length) as the variables which it is a function of, and this is not generally the case.

To avoid this pitfall, while retaining as much physical insight as possible, let us consider the temperature T in a room, as a known function of the (x,y,z) co-ordinate which defines our position in the room. For any specified values of x, y and z we can, given the form of the function, calculate T and its *partial derivatives* with respect to x, y and z: in other words, the rate at which T will vary per unit distance travelled along the X, Y and Z directions, respectively.

If we were to make a small move of length δx in the X direction from our starting point, T would vary by the amount:

$$\Delta T = \frac{\partial T}{\partial x}\delta x \qquad 1.1$$

If we go too far in the X direction, this estimate for ΔT becomes progressively less accurate since it is based on the value for $\frac{\partial T}{\partial x}$ which was calculated at the starting point. However, as δx becomes infinitesimally small, this expression becomes increasingly accurate.

Also, provided δx is small, the values of $\frac{\partial T}{\partial y}$ and $\frac{\partial T}{\partial z}$ which were calculated at the start point are still reasonably accurate: so if we now move a small distance δy in the Y direction followed by δz in the Z direction, the total change in temperature will be:

$$\Delta T = \frac{\partial T}{\partial x}\delta x + \frac{\partial T}{\partial y}\delta y + \frac{\partial T}{\partial z}\delta z \qquad 1.2$$

This can be rewritten as if it were the dot product of two vectors:

$$\Delta T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right) \cdot (\delta x , \delta y , \delta z) \qquad 1.3$$

where $(\delta x , \delta y , \delta z)$ is indeed the vector we have travelled along from our starting point, and

$$\left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right) = \nabla T \qquad 1.4$$

(usually known as Grad T) is a vector whose direction and magnitude we can calculate, given the formula for T and the starting point for the move.

Suppose, then, that we are allowed to move 1 mm from our given start point, and have been told to go in the direction which will cause the greatest rise in temperature: how can we calculate which direction to go in? Well, ∇T has been completely defined, and the length of $(\delta x , \delta y , \delta z)$ has been fixed; and we recall that the dot product of two vectors is the product of their magnitudes times the cosine of the angle between them. Since we can't alter the magnitude of $(\delta x , \delta y , \delta z)$, the best thing we can do is to point it in the same direction as ∇T.

We can thus see that Grad T is a vector which actually points in the direction of maximum change of T, for every point where it is calculated. Moreover, its magnitude tells us how much T will alter per unit distance travelled in that direction.

What if we wanted to go in a direction which caused no change in T at all? We can simply go in any direction perpendicular to the direction in which Grad T is pointing. Of course once we move away from our starting point the direction of Grad T will alter a bit, so we may have to alter our course to keep T constant; but if we are careful, we can move about indefinitely on a curved, shell-like surface of constant T. In fact, the scalar field of T can be thought of as a whole *family of nested shells*, each having a different value of T, and with Grad T giving the direction of the surface normal of the patch of shell which passes through any point in space; the shells can be closely packed (in areas where the magnitude of Grad T is large), but provided the function for T is continuous, they will never actually pass through each other.

If we were to set up a planar surface cutting diagonally across our room, it would cut through a number of these shells, and we could plot curved lines on the surface to mark where each shell passed through it. These would be contour lines of constant T, exactly as described at the beginning of this section.

2 Vector Fields and Flux

Just as a Scalar Field is a scalar quantity which is a function of position, so a Vector Field is a vector quantity which is a function of position, e.g. the velocity of water flowing steadily through some passageway. A Vector Field in 3-dimensional space simply contains three separate scalar functions which control the (i, j, k) components of the vector:

$$\underline{U} = (f_1(x,y,z) , f_2(x,y,z) , f_3(x,y,z)) \qquad 2.1$$

An important concept in Vector Fields is the amount of *vector flux* which flows through a small planar area fixed in the space where the field exists. In the case of our water velocity vector above, the vector flux would be the volume flowrate (in m^3/sec) through a small planar ring (think of the device used for blowing soap bubbles) held up in the path of its flow.

A useful way of representing such a patch of surface is in terms of its surface normal vector, i.e. a vector perpendicular to the surface whose magnitude represents the area of the patch. If we built a small rectangular bubble-blower and held it parallel to the XY plane such that it measured δx in the X direction and δy in the Y direction, it would be represented by:

$$\underline{da} = (0, 0, \delta x \times \delta y) \qquad 2.2$$

Suppose such a patch was held with its centre of area at a given point (x*, y*, z*) in our Vector Field: what would be the volume flowrate through it? It is clear that we are only interested in the $\underline{k}$ component of the water's velocity, since the other two components carry the water in a direction which is parallel to the surface. The velocity of the water through the patch at its centre is given by $f_3(x^*, y^*, z^*)$. If we went to the corner of the patch with the biggest (x,y) co-ordinates, we would find the $\underline{k}$ component of the velocity was slightly different: a first estimate of its value would be:

$$f_3(x^*, y^*, z^*) + \frac{\partial(f_3)}{\partial x}\frac{\delta x}{2} + \frac{\partial(f_3)}{\partial y}\frac{\delta y}{2} \qquad 2.3$$

But this change in velocity would be exactly matched by the equal and opposite change we would measure if we went to the opposite corner of the patch; and the same argument can be applied to any other point in the patch. So it is reasonable to say that the volume flowrate of water through the patch is

$$f_3(x^*, y^*, z^*) \times \delta x \times \delta y \qquad 2.4$$

provided that the partial derivatives of f_3 do not vary significantly within the patch. This can be seen to be the same as

$$\underline{U} \cdot \underline{da} \qquad 2.5$$

using the definitions given in (2.1) and (2.2), and this is indeed the expression which gives the amount of vector flux through any patch of surface held at any orientation in the space where $\underline{U}$ exists. Rather like Grad T, therefore, $\underline{U}$ can be seen as giving us the direction we should orientate our patch in for maximum flux per unit area; and its magnitude tells us the flux per unit area which we will then obtain.

3 Div

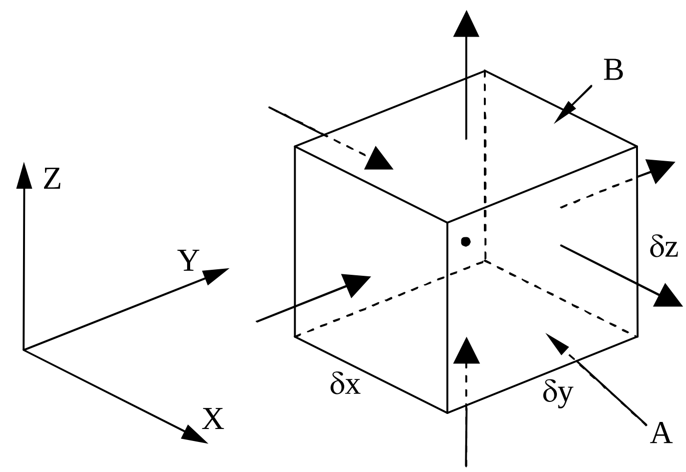
Go out into the space where $\underline{U}$ exists, and build a rectangular wire-frame box which has all its edges parallel to the X Y and Z axes, and which measures $\delta x \times \delta y \times \delta y$. Position it such that the point (x*, y*, z*) is at its centre.

The box has two faces parallel to the XY plane: face A, which is centred at (x*, y*, z*- δz/2), and face B which is centred at (x*, y*, z*+δz/2). From (2.4) we can say that the volume flowrate of water *into* the box through face A is

$$f_3(x^*, y^*, z^* - \delta z/2)\ (\delta x \times \delta y) \qquad 3.1$$

which can be approximated by

$$(f_3(x^*, y^*, z^*) - \frac{\partial(f_3)}{\partial z}\frac{\delta z}{2})\ \ (\delta x \times \delta y) \qquad 3.2$$



Likewise, the volume flowrate of water *out of* the box through face B is

$$(f_3(x^*, y^*, z^*) + \frac{\partial(f_3)}{\partial z}\frac{\delta z}{2})\ \ (\delta x \times \delta y) \qquad 3.3$$

Subtracting (3.2) from (3.3) gives the *net outflow* of water through these two faces, i.e.:

$$(\frac{\partial(f_3)}{\partial z}\delta z)\ \ (\delta x \times \delta y) \qquad 3.4$$

Similarly, we can calculate the net outflow of water through the two faces parallel to the YZ plane to be:

$$(\frac{\partial(f_1)}{\partial x}\delta x)\ \ (\delta y \times \delta z) \qquad 3.5$$

and that for the two faces parallel to the ZX plane to be:

$$(\frac{\partial(f_2)}{\partial y}\delta y)\ \ (\delta x \times \delta z) \qquad 3.6$$

Adding (3.4), (3.5) and (3.6) gives the overall net efflux for the whole box, i.e.:

$$(\frac{\partial(f_1)}{\partial x} + \frac{\partial(f_2)}{\partial y} + \frac{\partial(f_3)}{\partial z})\ \delta x\ \delta y\ \delta z \qquad 3.7$$

Since $\delta x\ \delta y\ \delta z$ is the volume of the box, we can define $(\frac{\partial(f_1)}{\partial x} + \frac{\partial(f_2)}{\partial y} + \frac{\partial(f_3)}{\partial z})$ to be the *net efflux per unit volume* of the space close to the centre of the box. We can think of this expression as a sort of dot product too, i.e.:

$$(\frac{\partial(f_1)}{\partial x} + \frac{\partial(f_2)}{\partial y} + \frac{\partial(f_3)}{\partial z}) = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})\ .\ (f_1, f_2, f_3) = \nabla\ .\ \underline{U} \qquad 3.8$$

which is known as Div U. ∇ (known as Del, or sometimes as Nabla) can thus be seen to be a kind of *vector differential operator*: in (1.4) it is shown acting on a scalar function (akin to multiplying a vector by a scalar) and in (3.8) it is shown being ‘dotted’ with a vector function.

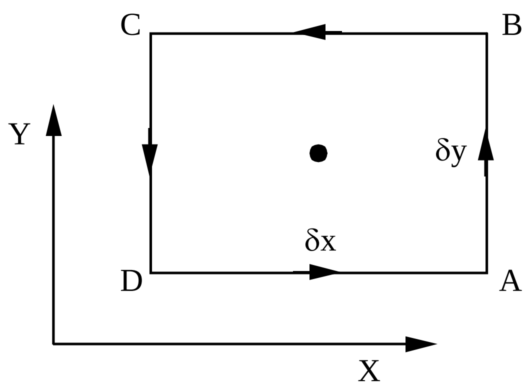
So what’s the significance of ‘net efflux per unit volume’? Well in the case of a vector field such as $\underline{U}$, which represents the velocity of water in steady flow, we would normally expect to find that Div $\underline{U}$ is everywhere zero. Such a Vector Field is said to be Solenoidal. Sometimes, velocity fields for incompressible fluids are set up with a ‘source’ or a ‘sink’ at some point in the field: it is likely that Div $\underline{U}$ will be infinite at such a point, representing an infinite efflux per unit volume existing over an infinitesimally small volume. For compressible fluids in steady flow, it is permissible for Div $\underline{U}$ to be non-zero - but Div (ρ $\underline{U}$) will be zero, where ρ, the density, is a function of position (i.e. a Scalar Field).

4 Curl

If Del can be dotted with a Vector Field to produce Div, can we find a use for the cross product of Del and the Vector Field as well? The answer is yes, and it’s called Curl.

To see how Curl works, let us think of a Vector Field of forces, i.e.

$$\underline{F} = (f_1(x,y,z)\,,\, f_2(x,y,z)\,,\, f_3(x,y,z)\,) \qquad 4.1$$



In some force fields, it is possible to extract mechanical work by allowing a particle (upon which the forces act) to travel round in some kind of closed loop, arriving back at its starting point. Suppose we constructed a small rectangular planar patch of size $\delta x \times \delta y$, parallel to the XY plane and centred on the point (x^*, y^*, z^*). Travelling from corner A to corner B, the mean force along the direction of motion will be

$$f_2(x^* + \delta x/2,\, y^*,\, z^*) \approx f_2(x^*, y^*, z^*) + \frac{\partial(f_2)}{\partial x}\,\delta x/2 \qquad 4.2$$

i.e. the force in the $\underline{j}$ direction at the midpoint of the side AB. The work *extracted* in allowing the particle to travel from A to B less the work *required* to move it from C to D can thus be calculated to be

$$\{f_2(x^*, y^*, z^*) + \frac{\partial(f_2)}{\partial x}\,\delta x/2\}\delta y - \{f_2(x^*, y^*, z^*) - \frac{\partial(f_2)}{\partial x}\,\delta x/2\}\delta y = \{\frac{\partial(f_2)}{\partial x}\,\delta x\}\,\delta y \qquad 4.3$$

Likewise, the work extracted going from B to C minus the work required to go from D to A is

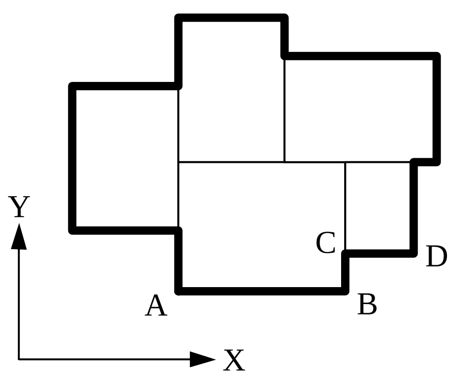
$$\{-\frac{\partial(f_1)}{\partial y}\delta y\}\ \delta x \qquad 4.4$$

Adding (4.3) and (4.4) gives the net work we will extract if the particle makes a complete tour round the boundary of the patch in an anticlockwise direction:

$$\{\frac{\partial(f_2)}{\partial x} - \frac{\partial(f_1)}{\partial y}\}\ \delta x\ \delta y \qquad 4.5$$

Since $\delta x\ \delta y$ is the area of the rectangle in the XY plane, it can be seen that the quantity $(\frac{\partial(f_2)}{\partial x} - \frac{\partial(f_1)}{\partial y})$ is the *net work per unit area* which can be extracted by going anticlockwise round the edge of the area.

Now imagine creating a small non-rectangular patch (such that the quantity $\frac{\partial(f_2)}{\partial x} - \frac{\partial(f_1)}{\partial y}$ is constant over the whole patch) out of several even smaller rectangles, as shown below.



From equation 4.5, the work extracted by going anticlockwise around each rectangle in turn would be $(\frac{\partial(f_2)}{\partial x} - \frac{\partial(f_1)}{\partial y})$ multiplied by the total area of the patch. But in doing this, each thin line is passed along exactly twice, in opposite directions. Therefore, the same work would also be extracted by simply going around the outside perimeter of the patch (thick lines, visiting A, B, C, &c.). In other words, $(\frac{\partial(f_2)}{\partial x} - \frac{\partial(f_1)}{\partial y})$ multiplied by the area of the patch always gives the net work no matter what shape the patch is (so long as it is small enough for $(\frac{\partial(f_2)}{\partial x} - \frac{\partial(f_1)}{\partial y})$ to be taken as a constant over the whole patch).

Now imagine that what we are seeing when we look down on the XY plane is actually the projection of a planar path floating in space, which is not parallel to any of the principal planes. We could equally look at this path along the X axis, where we would see it projected onto the YZ plane; and if we looked down along the Y axis we would see it projected onto the ZX plane. Now, as the particle makes the trip round the patch from A to B, and so on round to A again, it picks up the extra work terms which arise from the fact that it is also moving around in the YZ plane, and in the ZX plane. Using the method explained above, these extra terms can be shown to be:

$$\{ \frac{\partial(f_3)}{\partial y} - \frac{\partial(f_2)}{\partial z} \} \, \delta A_{YZ} \; + \; \{ \frac{\partial(f_1)}{\partial z} - \frac{\partial(f_3)}{\partial x} \} \, \delta A_{ZX} \qquad 4.6$$

where δA_{YZ} and δA_{ZX} are the areas projected by the planar path onto the YZ and ZX planes respectively.

How would the surface area and orientation of this planar path be represented vectorially, using the method described in Section 2? Using the same convention, we would write:

$$\underline{\delta a} = (\delta A_{YZ}, \delta A_{ZX}, \delta A_{XY}) \qquad 4.7$$

so that the sum of (4.5) and (4.6) can be written:

$$(\{ \frac{\partial(f_3)}{\partial y} - \frac{\partial(f_2)}{\partial z} \} , \{ \frac{\partial(f_1)}{\partial z} - \frac{\partial(f_3)}{\partial x} \} , \{ \frac{\partial(f_2)}{\partial x} - \frac{\partial(f_1)}{\partial y} \}) \cdot \underline{\delta a} \qquad 4.8$$

The first vector in this dot product, which is what we mean by Curl F, is obtained by taking the 'cross product' of ∇ with F, i.e.

$$(\frac{\partial}{\partial x} , \frac{\partial}{\partial y} , \frac{\partial}{\partial z}) \times (f_1 , f_2 , f_3) = \nabla \times \underline{F} \qquad 4.9$$

So $(\nabla \times \underline{F}) \cdot \underline{\delta a}$ tells us how much work we will extract from the force field if we let our particle travel round the boundary of our patch of surface. Similar to Grad, then, the *direction* of this Curl vector tells us how we should orientate our patch for maximum work output - and its *magnitude* tells us how much work we will then extract per unit area of patch. Remember that the patch does not have to be square, or rectangular - it is the position, area and orientation of the patch which matter, not its shape.

Another useful way of looking at the expression in (4.8) is to think of it as measuring the amount of 'Curl flux' passing through the patch of surface, just as in (2.5).

5 Gauss' Theorem

Gauss' Theorem is quite straightforward - it simply points out that there are two ways of measuring the net amount of vector flux coming out of a defined volume of space in a Vector Field, and that they will both give the same result.

The first way is to break the surface covering the volume down into small patches with their surface normals pointing outwards, and measure the amount of flux coming out of each patch, using (2.5). If all the results are then summed up, this will give the net amount of flux coming out of the volume, since any patch which has flux going into the volume through it will show up as a negative quantity. The mathematical way of expressing this operation is:

$$\iint_{\text{Surface}} \underline{U} \cdot \underline{da} \qquad 5.1$$

A double integral is used because two parameters are needed to define each position on a surface - e.g. latitude and longitude for positions on the surface of the Earth.

The second way is to break the volume down into small elemental volumes, and measure the net amount of flux produced by each one using Div. Again this will sum to give the net amount of flux produced by the total volume, since any elemental volume which 'swallows' flux will show up as a negative quantity. The mathematical expression for this operation is:

$$\iiint_{\text{Volume}} (\nabla . \underline{U}) dv \qquad 5.2$$

and Gauss' Theorem simply says that (5.1) and (5.2) are equal.

6 Stokes' Theorem

What Gauss is to Div, Stokes is to Curl. Stokes' Theorem provides two ways of measuring the work which can be extracted when a particle moves round a closed loop (not necessarily small or even planar) in a force field, and says they will both give the same result.

The first way is to break down the path into small elements $\underline{ds}$, and take the dot product of each one with $\underline{F}$, to measure the amount of work extracted as the particle moves along that element. The total work is found by summing these round the loop, i.e.

$$\oint_{\text{Loop}} \underline{F} . \underline{ds} \qquad 6.1$$

The second way is to construct a surface (of any shape) upon which the loop lies, throw away the parts outside the loop, and divide the part inside the loop into small patches $\underline{da}$. The work extracted by going round the perimeter of one of these patches is $(\nabla \times \underline{F}) \,.\, \underline{da}$, as explained in section 4.

Now consider two neighbouring patches of surface, $\underline{da}_1$ and $\underline{da}_2$. Like neighbouring pages of a half-opened book, they are not necessarily coplanar, but they do have a boundary in common. Travelling anticlockwise round their combined outer boundary will yield exactly the same amount of work as travelling anticlockwise round the two individual patches separately: the only difference between these two operations is that the second one involves moving along the common boundary twice, once in one direction and once in the other, which clearly has no effect on the work extracted (just as in Section 4). Extending this argument to all the patches we have just created, and using the final result of Section 4, tells us that the total work extracted by going round the loop will be:

$$\iint_{\text{Surface}} (\nabla \times \underline{F}) . \underline{da} \qquad 6.2$$

and Stokes' Theorem is that (6.1) and (6.2) are equal.

7 Curl of Grad, and Scalar Potential

The Grad vectors which can be calculated at each point in a Scalar Field clearly form a Vector Field in their own right. We can therefore calculate Curl(Grad T) - but why should we want to?

Let us think of a slightly different sort of Scalar Field - the field P of potential energies which we would assign to a 1kg mass at different places in the earth's gravitational field. Under these circumstances Grad P tells us, at each point, the direction we should move the mass for maximum increase in potential energy, and the amount by which the potential energy would increase per unit distance travelled. In other words, Grad P is a Vector Field which is always exactly equal and opposite to the field of gravitational forces $\underline{F}$ which acts on the mass.

It is clear that no work can ever be extracted by moving a mass round a closed loop in a field of forces if it possible to assign definite potential energies to every point in the field: since the mass will have the same potential energy at the end of the loop as when it started, no net work can have been extracted. We can therefore predict that the Curl of such a Vector Field will be zero everywhere, and so establish the identity:

$$\nabla \times (\nabla P) = \underline{0} \qquad 7.1$$

for *any* scalar field P. It further follows that if we come across a Vector Field whose Curl is always zero, that field has a set of scalar potentials, *of which it is the Grad*. Such a Vector Field is said to have *Scalar Potential*, and the necessary and sufficient test for a Vector Field to have Scalar Potential is that its Curl should be everywhere zero.

8 Div of Curl, and Vector Potential

The Curl of a Vector Field is itself another Vector Field, so it is mathematically legitimate to calculate its Div - but again, it seems a bit pointless at first sight.

In section 6, we saw how the amount of work available from travelling around a closed (but not necessarily planar) loop could be calculated by constructing a surface of which the loop formed a border, and measuring the amount of Curl flux which passed through it. But we didn't have to specify the exact shape of the surface - any surface which fitted onto the loop all the way round its edge would have done, and the Curl flux through the surface would still have been a measure of the work available from the loop.

So let us fit two different surfaces to the loop, such that there is a volume of space contained between them: if the amount of Curl Flux passing through each surface is the same, there cannot be any net generation of flux in the enclosed volume. In other words, the Div of a Curl Field must be everywhere zero. We can therefore establish the identity

$$\nabla \cdot (\nabla \times \underline{F}) = \underline{0} \qquad 8.1$$

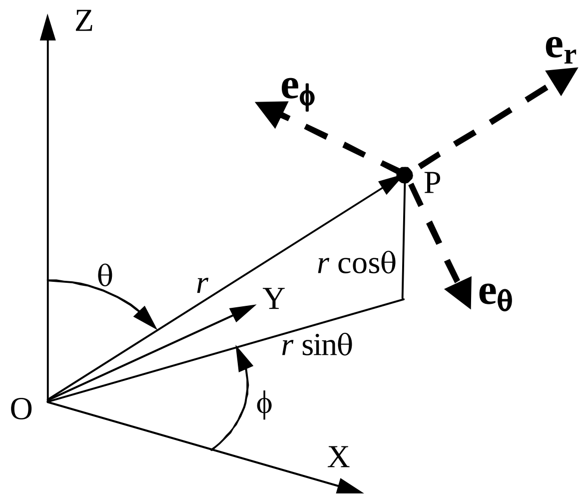
for any Vector Field $\underline{F}$. It is thus the case that any Vector Field which is Solenoidal (i.e. in which Div is everywhere zero) can be thought of as being the Curl field of some other Vector Field. Such a Vector Field is then said to have *Vector Potential*, and the necessary and sufficient test for a Vector Field to have Vector Potential is that its Div should everywhere be zero.

Appendix A: Other co-ordinate systems

The discussion in this guide has been based on a Cartesian (Y,Y,Z) co-ordinate system, which helps keeps thing simple – for instance, an elemental volume is cuboidal in shape, its volume is ($\delta x \times \delta y \times \delta z$), and its opposite faces have the same area as each other.

Sometimes, it is more convenient to use other co-ordinate systems, for instance cylindrical polar, or spherical polar (e.g. for atmospheric temperatures round the surface of the Earth). All the concepts developed in the main text still work – but the equations become more complex, because the small movements and elemental volumes are no longer so simple.

For instance, the derivation of Grad and Div in spherical polar co-ordinates are as follows:

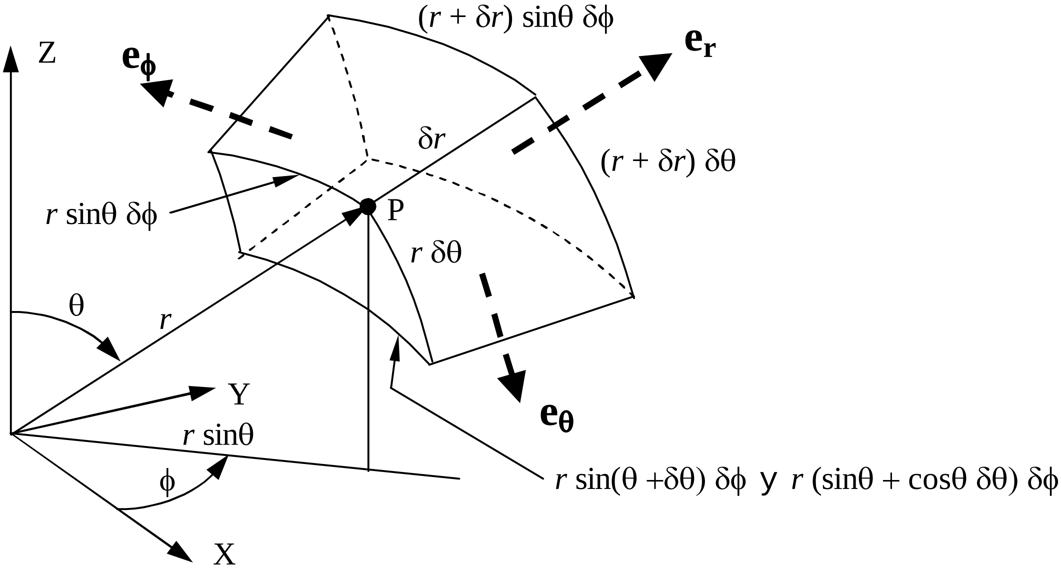


The co-ordinates of a point P in such a system are (r,θ,ϕ), as shown above; r is the distance of P from the origin, θ is the angle that OP makes with the Z axis, and ϕ is the angle between the X axis and the projection of OP onto the XY plane. There are three associated orthogonal vectors, $\mathbf{e}_r$ $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$, which are unit vectors in the directions that P would travel in if r, θ or ϕ were increased by a small amount.

The definition of Grad in equation (1.4) can be thought of as an (**i**,**j**,**k**) vector, whose scalar components are the amounts by which the value of the scalar field T would increase if a (small) unit step was made in the **i**, **j** and **k** directions respectively. In the spherical polar system, we therefore want the increases in T when P moves by one (small) unit in the $\mathbf{e}_r$ $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ directions.

In the case of the $\mathbf{e}_r$ direction, this is simple: when r increases by δr, P moves by δr in the $\mathbf{e}_r$ direction, so the increase per unit movement is simply $\frac{\partial T}{\partial r}$. But when θ increases by $\delta\theta$, P moves by ($r\ \delta\theta$) in the $\mathbf{e}_\theta$ direction; and when ϕ increases by $\delta\phi$, P moves by ($r \sin\theta\ \delta\phi$) in the $\mathbf{e}_\phi$ direction. To get the increases per *unit* movement in those directions, we must therefore define the Grad of T as follows:

$$\nabla T \ = \ \frac{\partial T}{\partial r}\mathbf{e}_r \ + \ \frac{1}{r}\frac{\partial T}{\partial \theta}\mathbf{e}_\theta \ + \ \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\mathbf{e}_\phi \qquad \text{A.1}$$



For Div, we first note that the volume V of a small space near to P is given by:

$$V = \delta r \times r\delta\theta \times r \sin\theta\, \delta\phi = r^2 \sin\theta\, \delta r\, \delta\theta\, \delta\phi \qquad \text{A.2}$$

A vector field $\underline{U}$ can be defined (c.f. equation 2.1) as:

$$\underline{U} = f_r(r,\theta,\phi)\,\mathbf{e}_r + f_\theta(r,\theta,\phi)\,\mathbf{e}_\theta + f_\phi(r,\theta,\phi)\,\mathbf{e}_\phi \qquad \text{A.3}$$

The net efflux of $\underline{U}$ in the $\mathbf{e}_r$ direction (i.e. through the two faces perpendicular to the $\mathbf{e}_r$ direction) is given by the flux *out* of the volume through the far face minus the flux *into* the volume through the near face, i.e.:

$$(f_r + \frac{\partial(f_r)}{\partial r}\delta r) \times (r + \delta r)\,\delta\theta \times (r + \delta r)\sin\theta\,\delta\phi \;-\; f_r \times r\,\delta\theta \times r\sin\theta\,\delta\phi$$

When multiplied out fully , this expression yields terms in δ^2, δ^3, δ^4 and δ^5. The terms in δ^2 all cancel out, and the terms in δ^4 and δ^5 can be ignored, because they are much smaller than those in δ^3. The net efflux then becomes

$$\frac{\partial(f_r)}{\partial r}\delta r \times r\,\delta\theta \times r\sin\theta\,\delta\phi + f_r \times (r\,\delta\theta \times \delta r\sin\theta\,\delta\phi + \delta r\,\delta\theta \times r\sin\theta\,\delta\phi)$$

$$= \frac{V}{r^2}\left(r^2\frac{\partial(f_r)}{\partial r} + 2r\,f_r\right) = \frac{V}{r^2}\frac{\partial}{\partial r}\left(r^2\,f_r\right) \qquad \text{A.4}$$

using (A.2).

The net efflux in the $\underline{e}_\theta$ direction is given by a similar expression:

$$(f_\theta + \frac{\partial(f_\theta)}{\partial\theta}\delta\theta) \times \delta r \times r\,(\sin\theta + \cos\theta\,\delta\theta)\,\delta\phi - f_\theta \times \delta r \times r\sin\theta\,\delta\phi$$

which likewise simplifies down to

$$\frac{\partial(\mathrm{f}_\theta)}{\partial\theta}\,\delta\theta \times \delta r \times\; r \sin\theta\;\delta\phi + \;\mathrm{f}_\theta \times \delta r\; \times r\cos\theta\;\delta\theta\;\delta\phi$$

$$= \;\frac{V}{r\sin\theta}\left(\sin\theta\frac{\partial(\mathrm{f}_\theta)}{\partial\theta} + \cos\theta\,\mathrm{f}_\theta\right) \;=\; \frac{V}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\mathrm{f}_\theta\right) \qquad \text{A.5}$$

The net efflux in the $\underline{e}_\phi$ direction is given by a much simpler expression, because the two faces perpendicular to this direction both have the same area:

$$\frac{\partial(\mathrm{f}_\phi)}{\partial\phi}\,\delta\phi \times \delta r \times r\;\delta\theta \;=\; \frac{V}{r\sin\theta}\frac{\partial(\mathrm{f}_\phi)}{\partial\phi} \qquad \text{A.6}$$

The total net efflux per unit volume is then obtained by adding the expressions from (A.4), (A.5) and (A.6) together and dividing through by *V*, to give:

$$\nabla\,.\,\underline{\mathrm{U}} \;=\; \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\mathrm{f}_\mathrm{r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\mathrm{f}_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial(\mathrm{f}_\phi)}{\partial\phi} \qquad \text{A.7}$$

Finally, we can build an expression for the ‘Div of Grad’, or ∇^2, by comparing equation (A.1) with (A.3), and then using it to substitute for f_r, f_θ and f_ϕ in (A.7):

$$\nabla^2\,\mathrm{T} \;=\; \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial\mathrm{T}}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\mathrm{T}}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\mathrm{T}}{\partial\phi^2} \qquad \text{A.8}$$