RESEARCH NOTE: DEFLECTION EQUATION FOR THE BUCKLING OF AN ELASTIC COLUMN SUBJECTED TO SURFACE PRESSURE

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The deflection equation for the buckling of an initially straight elastic column subjected to external or internal pressure is derived for the case when the pressure and the area of the column may vary along its length. Apparently this equation has not been reported in the literature previously.

Introduction

THE EFFECT of pressure on the buckling of elastic columns has been studied for at least the last thirty years. Haringx considered the instability of thin-walled cylinders under internal pressure (1)† in his analysis of the behaviour of corrugated bellows expansion joints under internal pressure. Nagel (2) and Peterson (3) were concerned with aerospace applications where slender columns are subjected to axial load and lateral pressure. Other authors have considered the theoretical aspects (Handelman (4)) and reported the results of experimental tests (Mills (5)).

The deflection equation used by these authors (see, for instance, equation (1) of (3)) is

$$EI\frac{d^4y}{dx^4}+(P-pA)\frac{d^2y}{dx^2} = 0 \quad . \quad . \quad (1)$$

where y is the lateral deflection of the neutral axis of an initially straight elastic column of constant external cross-sectional area A (symmetrical about the neutral plane) and bending stiffness EI when subjected to an external (compressive) pressure p at section x and a central compressive end load P. The same equation applies for a hollow tube subjected to internal pressure provided that P is still the compressive load applied directly to the ends of the tube, A is now the internal cross-sectional area, and p becomes the hydrostatic tension (negative pressure) applied to the internal walls. All but one of the authors considered only the case when the applied pressure is constant along the length of the column. However, Handelman (4) considered a pressure distribution varying linearly from one end of the column to the other end.

The purpose of this note is to show that equation (1) is only correct when the applied pressure and area are constant along the length of the column, and that, when pressure and area vary the deflection equation should be

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right)+(P-pA)\frac{d^2y}{dx^2}-A\frac{dp}{dx}\frac{dy}{dx} = 0 \quad (2)$$

where P is now the axial compressive load in the column at section x. The pressure $p(x)$ is assumed to be a function of x only so that surfaces of constant pressure are perpendicular to the axis of the straight undeflected column.

This equation, which is derived below, is shown to produce results in agreement with those obtained by the author by another method when studying the effect of varying external pressure on the instability of a cantilever beam (6). It is believed that the results given in Handelman's paper (4) require amendment to take account of the term $-A(dp/dx)(dy/dx)$ which occurs in equation (2) but is absent from equation (1).

Derivation of the deflection equation

The column will be assumed to be a 'slender member' for which Euler's equation

$$M = EI\frac{d^2y}{dx^2} \quad . \quad . \quad . \quad . \quad (3)$$

relates the applied bending moment M at section x to the (approximate) curvature d^2y/dx^2. Consider the equilibrium of a small element of the column of length dx (Fig. 1). For convenience the column is assumed to have a rectangular section of width b (constant) and depth $2h(x)$. The internal stresses acting on the cut sections of the column element are represented by the resultant axial force P, shear force V and moment M, as shown. The externally applied lateral pressure is taken to be locally hydrostatic so that the pressure forces are always normal to the external surface. For equilibrium of the element shown in Fig. 1, resolving forces axially,

$$\frac{dP}{dx} = p\frac{dA}{dx} \quad . \quad . \quad . \quad . \quad (4)$$

resolving forces laterally,

$$\frac{dV}{dx} = 0 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

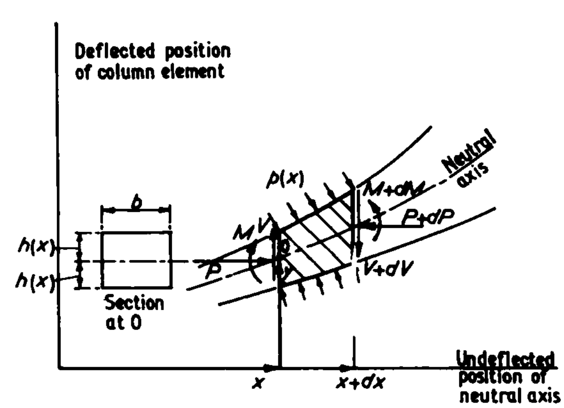

Fig. 1

The MS. of this research note was received at the Institution on 14th April 1972 and accepted for publication on 8th August 1972. 2

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and taking moments about 0

$$\mathrm{d}M+P\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x-V\,\mathrm{d}x-pb\left(\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x\right)(2h)=0$$

or

$$\frac{\mathrm{d}M}{\mathrm{d}x}+P\frac{\mathrm{d}y}{\mathrm{d}x}-V-pA\frac{\mathrm{d}y}{\mathrm{d}x}=0 \quad . \quad . \quad (6)$$

Differentiating equation (6) to eliminate the unknown shear force V gives

$$\frac{\mathrm{d}^2M}{\mathrm{d}x^2}+\frac{\mathrm{d}P}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x}+P\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\frac{\mathrm{d}}{\mathrm{d}x}\left(pA\frac{\mathrm{d}y}{\mathrm{d}x}\right)=0$$

The $\mathrm{d}P/\mathrm{d}x$ term may be eliminated from equation (4), and using equation (3) then gives

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+p\frac{\mathrm{d}A}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x}+P\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-A\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}y}{\mathrm{d}x}\right)-p\frac{\mathrm{d}A}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

Finally, eliminating like terms of opposite sign, the deflection equation becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+(P-pA)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-A\frac{\mathrm{d}p}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

as already stated.

This analysis may be extended to the case of a column of non-rectangular section by considering the equilibrium of a thin, parallel sided slice through the non-rectangular column element, and then summing the results for all the slices of the element to verify equations (4)–(6). Alternatively, the following independent approach may be used.

Alternative derivation

In (**6**) the deflection equation for a cantilever beam subjected to external pressure has been obtained directly in the integrated form (equation (4) of (**6**))

$$EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\int_x^l\left(A\frac{\mathrm{d}p}{\mathrm{d}x}\right)_{x=\xi}(\eta-y)\,\mathrm{d}\xi=0 \quad . \quad (7)$$

where y is the deflection at section x and η is the deflection at section ξ (Fig. 2). The differential deflection equation (2) may also be obtained by differentiating this equation twice.

The standard formula (7) for differentiating an integral involving a variable parameter (in this case x) can be used, noting that $\left(A\frac{\mathrm{d}p}{\mathrm{d}x}\right)_{x=\xi}$ is here a function of ξ so that its partial derivative with respect to the limiting value x is zero.

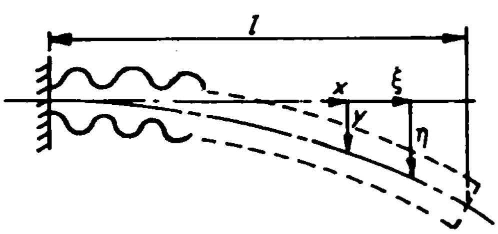

Fig. 2

According to the formula, differentiating equation (7) once gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+\frac{\mathrm{d}y}{\mathrm{d}x}\int_x^l\left(A\frac{\mathrm{d}p}{\mathrm{d}x}\right)_{x=\xi}\mathrm{d}\xi=0$$

and then differentiating again gives

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\int_x^l\left(A\frac{\mathrm{d}p}{\mathrm{d}x}\right)_{x=\xi}\mathrm{d}\xi-A\frac{\mathrm{d}p}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x}=0 \quad (8)$$

From the theory given in reference (**6**), the integral in equation (8) may be related to the axial force at section x. A 'fluid column' will only be in equilibrium if there is a distributed body force $\left(A\frac{\mathrm{d}p}{\mathrm{d}x}\right)\mathrm{d}x$ acting to the right (in Fig. 2). The axial force in the column at section x is then just

$$P=pA$$

However in the absence of such body forces, the axial compressive force at x must be

$$P=pA+\int_x^l\left(A\frac{\mathrm{d}p}{\mathrm{d}x}\right)_{x=\xi}\mathrm{d}\xi \quad . \quad . \quad (9)$$

in order to hold the column in place against the pressure forces pushing it to the left. This heuristic argument is explained more fully in (**6**) and may be verified mathematically by the methods described in the reference.

Substituting from equation (9) into equation (8) then gives

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+(P-pA)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-A\frac{\mathrm{d}p}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

thus confirming the previous derivation.

Example

Consider the case of a straight cantilever beam of constant cross-sectional area A and length l subjected to a pressure field varying linearly from zero at the fixed end to p_0 at the free end (Fig. 3).

The pressure at section x is $p=p_0\frac{x}{l}$ and the axial force $P=p_0A$, which, on substituting into (2), gives

$$EI\frac{\mathrm{d}^4y}{\mathrm{d}x^4}+p_0A\left(1-\frac{x}{l}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\frac{p_0A}{l}\frac{\mathrm{d}y}{\mathrm{d}x}=0\,. \quad (10)$$

In (**6**) it has been shown that this problem is identical with the problem of a vertical flagpole deflecting under its

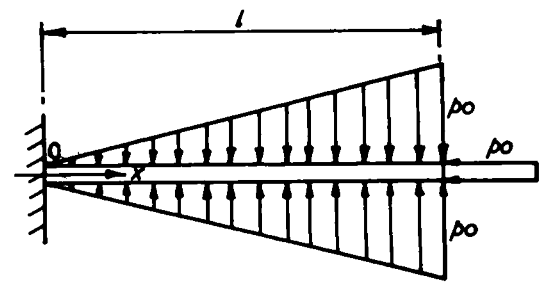

Fig. 3

own weight when the weight per unit length of the pole is $q = p_0 A/l$. The deflection equation for a flagpole is, from (8),

$$EI\frac{d^3y}{dx^3}+q(l-x)\frac{dy}{dx} = 0 \quad . \quad . \quad (11)$$

Putting $q = p_0 A/l$ and differentiating equation (11) gives

$$EI\frac{d^4y}{dx^4}+p_0 A\left(1-\frac{x}{l}\right)\frac{d^2y}{dx^2}-\frac{p_0 A}{l}\frac{dy}{dx} = 0$$

thus providing an independent check of equation (10).

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