Pedestrian excitation of bridges

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Abstract: This paper reviews the evidence on dynamic bridge loading caused by moving pedestrians. The phenomenon of 'synchronization' by which people respond naturally to an oscillating bridge when this has a frequency close to their natural walking or running frequency is a feature of this phenomenon. By increasing modal damping, synchronization can be prevented, but how much damping is needed in any particular situation?

If some simplifying assumptions about how people walk are made, it is possible to predict analytically the minimum damping required to ensure that synchronization does not lead to high vibration levels. The main assumption is that the movement of a pedestrian's centre of mass has two components. One is its natural movement when the person is walking on a stationary pavement. The other is caused by movement of the pavement (or bridge) and is in proportion to pavement amplitude but with a time delay that is arbitrary. When the time delay is a 'worst case', pedestrians act as a source of negative damping.

This theory supports the adoption of a non-dimensional number which measures the susceptibility of a bridge to pedestrian excitation. Although currently there are not many good bridge response data, predictions using this non-dimensional number are compared with the data that are available and found to be in satisfactory agreement. Both lateral and vertical vibrations are considered.

Keywords: bridge, vibration, pedestrian, synchronization, self-excitation, pedestrian loading, pedestrian excitation, bridge dynamics, Scruton number, pedestrian Scruton number, London Millennium bridge

NOTATION

f_n	natural frequency (Hz)
$f(t)$	modal force (alternatively, force per unit length of pavement, depending on the context)
$f_0(t)$	force per person exerted on pavement
k	lateral force per person/amplitude of pavement lateral velocity
K	modal stiffness of bridge
m	modal mass of pedestrians for the relevant bridge mode (alternatively, mass of pedestrians per unit length of bridge)
m_r	non-dimensional mass ratio $= \alpha\beta m/M$
m_0	mass per person
M	modal mass of bridge (alternatively, mass of bridge per unit length)
S_c	Scruton number
S_{c_p}	pedestrian Scruton number
t	time
$x(t)$	natural displacement of a pedestrian's centre of mass when walking on a stationary pavement (alternatively, corresponding modal displacement for all the pedestrians on a bridge when normalized by the relevant bridge mode)
$X(i\omega)$	Fourier transform of $x(t)$
$y(t)$	displacement of the pavement (alternatively, modal displacement of the bridge deck for the relevant mode)
$Y(i\omega)$	Fourier transform of $y(t)$
$z(t)$	total displacement of a pedestrian's centre of mass
α	ratio of the amplitude of movement of a person's centre of mass to the amplitude of movement of the pavement
$\alpha y(t-\Delta)$	movement additional to $x(t)$ of a person's centre of mass caused by pavement movement (alternatively, relevant modal displacement)
β	correlation factor for when individual people's natural movement synchronize ($\beta = \gamma$ assumed)

The MS was received on 2 July 2003 and was accepted after revision for publication on 9 February 2004.

γ	correlation factor for when individual people synchronize with pavement movement
Δ	time lag between movement of pavement and movement of a person's centre of mass
ϕ	phase angle $= \omega\Delta$
ϕ_c	critical phase angle for the onset of instability
ω	(angular) frequency
ω_c	critical frequency at which instability first occurs
ω_n	natural frequency of the relevant mode
Ω	non-dimensional frequency ratio $= \omega/\omega_n$
ς	damping ratio
ς_c	critical damping ratio required for stability
ς_{eff}	effective modal damping ratio

1 INTRODUCTION

Since the pedestrian-excited vibration of the London Millennium Bridge in June 2000, there has been considerable interest in bridge vibrations caused by the movement of people. Extensive studies by the Arup Partnership concentrated on quantifying the excitation of the Millennium Bridge and devising a way of increasing the bridge's damping. This they achieved with complete success. By artificially adding damping to the bridge, they were able to extinguish the self-excitation mechanism and so eliminate a vibration that was sufficient to cause people to stop walking and hold on to the handrails and that was severe enough to give concern for the safety of less mobile walkers. Since that experience, there has been time to examine in more detail the mechanics of interaction between a pedestrian and a moving pavement and, in particular, the process of synchronization. This process causes people to fall into step with an oscillation and with each other to set up positive feedback, thereby causing an initially very small oscillation to build up. It will be shown below that, subject to necessary simplifying assumptions, a stability criterion can be expressed in terms of a dimensionless number involving the bridge's damping and the ratio of pedestrian mass to bridge mass. This is similar to the non-dimensional Scruton number used to quantify the susceptibility of a structure to wind-excited oscillations. The collection of more experimental data is needed, but the analysis here suggests a framework by which results can be compared on a logical basis. From the data currently available, a variety of existing bridges are examined using this approach, and it is found that bridges with a low pedestrian Scruton number are those bridges that are sensitive to pedestrian-excited vibrations.

2 BACKGROUND

A report in 1972 quoted by Bachmann and Ammann [**1**] in their International Association for Bridge and Structural Engineering book described how a new steel footbridge had experienced strong lateral vibration during an opening ceremony with 300–400 people. They explained how the lateral sway of a person's centre of gravity occurs at half the walking pace. Since the footbridge had a lowest lateral mode of about 1.1 Hz, the frequency of excitation was very close to the mean pacing rate of walking of about 2 Hz. Thus in this case ‘an almost resonating vibration occurred. Moreover it could be supposed that in this case the pedestrians synchronised their step with the bridge vibration, thereby enhancing the vibration considerably' (reference [**2**], p. 636). The problem is said to have been solved by the installation of horizontal tuned vibration absorbers.

A later paper by Fujino *et al.* [**3**] described observations of pedestrian-induced lateral vibration of a cable-stayed steel box girder bridge of similar size to the Millennium Bridge. It was found that, when a large number of people were crossing the bridge (2000 people on the bridge), lateral vibration of the bridge deck at 0.9 Hz could build up to an amplitude of 10 mm, while some of the supporting cables whose natural frequencies were close to 0.9 Hz vibrated with an amplitude of up to 300 mm. By analysing video recordings of pedestrians' head movement, Fujino *et al.* concluded that lateral deck movement encourages pedestrians to walk in step and that synchronization increases the human force and makes it resonate with the bridge deck. They summarized their findings as follows: ‘The growth process of the lateral vibration of the girder under the congested pedestrians can be explained as follows. First a small lateral motion is induced by the random lateral human walking forces, and walking of some pedestrians is synchronised to the girder motion. Then resonant force acts on the girder, consequently the girder motion is increased. Walking of more pedestrians are synchronised, increasing the lateral girder motion. In this sense, this vibration was a self-excited nature. Of course, because of adaptive nature of human being, the girder amplitude will not go to infinity and will reach a steady state.'

Enquiries subsequent to the opening of the London Millennium Bridge identified some other interesting examples of pedestrian-excited bridge vibration [**4**], including the surprising vibration of the Auckland Harbour Bridge in New Zealand. This is an eight-lane motorway bridge, with three separate parallel roadways.

In 1975, one roadway, with two traffic lanes, was closed to vehicles to allow a political march to pass over the bridge. Contemporary newsreel footage shows the large crowd walking in step as the roadway built up a large amplitude lateral vibration at about 0.6 Hz. This vibration was serious enough for stewards to go through the crowd calling for marchers to break step, when it subsided naturally. It is interesting that the marchers had not intended to march in step but had naturally fallen into step with each other, apparently after the bridge began to sway.

3 PEDESTRIAN LOADING DATA

The book by Bachmann and Ammann [1] discussed loading from human motions, distinguishing between walking, running, skipping and dancing. For walking and running, they pointed out that dynamic pavement load is dominated by the pacing frequency (Table 1).

Published data on dynamic loads are few, but Bachmann and Ammann quoted an example for a pedestrian walking at 2 Hz when the fundamental component (at 2 Hz) of vertical dynamic loading is 37 per cent of static weight and the fundamental component (at 1 Hz) of lateral dynamic loading is 4 per cent of static weight. In the vertical case, harmonics are less than about 30 per cent of the fundamental in amplitude (a typical load–time history is shown in Fig. 1); in the lateral case there may be a significant third harmonic and an example is quoted in which the third harmonic exceeds the lateral fundamental in amplitude.

These forces are for people walking on stationary pavements, but it was noted by Bachmann and Ammann that 'pedestrians walking initially with individual pace on a footbridge will try to adjust their step subconsciously to any vibration of the pavement. This phenomenon of feedback and synchronisation becomes more pronounced with larger vibration of the structure'. Also, for vertical vibration, Bachmann and Ammann noted that displacements of the order of 10–20 mm have to occur for the phenomenon to be noticeable, although they said that it is more pronounced for lateral vibrations. 'Presumably, the pedestrian, having noticed the lateral sway, attempts to re-establish his balance by moving his body in the opposite direction; the load he thereby exerts on the pavement, however, is directed so as to enhance the structural vibration.'

Table 1 Data on walking and running from Bachmann and Ammann [1]

	Pacing frequency (Hz)	Forward speed (m/s)	Stride length (m)	Vertical fundamental frequency (Hz)	Horizontal fundamental frequency (Hz)
Slow walk	1.7	1.1	0.60	1.7	0.85
Normal walk	2.0	1.5	0.75	2.0	1.0
Fast walk	2.3	2.2	1.00	2.3	1.15
Slow running (jogging)	2.5	3.3	1.30	2.5	1.25
Fast running (sprinting)	> 3.2	5.5	1.75	> 3.2	> 1.6

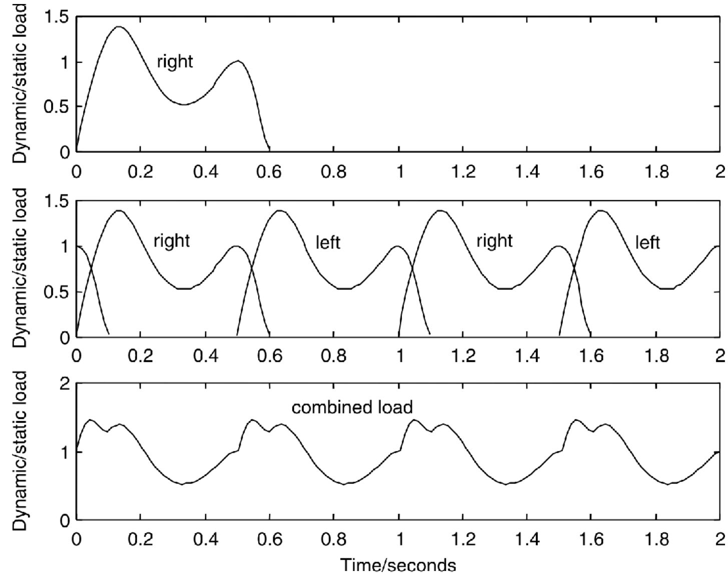

Fig. 1 Vertical load–time function from footfall overlap during walking at 1 pace/s. (After reference [1])

4 SYNCHRONIZATION

Fujino *et al.* [**3**] estimated from video recordings of crowd movement that some 20 per cent or more of pedestrians on their bridge were walking in synchronism with the bridge's lateral vibration, which had a frequency of about 0.9 Hz and an amplitude of about 10 mm. They computed the amplitude of steady state lateral vibration of this bridge, firstly, using the value of 23 N given by Bachmann and Ammann for the amplitude of lateral force per person and assuming that the pedestrians walk with random phase and, secondly, using a force per person of 35 N and the measured result that 20 per cent of them were synchronized to bridge movement. For the random phase case, the calculated amplitude is about 1 mm response; for the 20 per cent correlated case it is about 15 mm (compared with the measured value of 10 mm).

These results were thoroughly investigated following the London Millennium Bridge's problems, with the results given by Fitzpatrick *et al.* [**5**] (an amended version of this paper was subsequently published as reference [**4**]). Using moving platforms, data were measured on lateral dynamic force and on the probability that a pedestrian would synchronize with pavement lateral vibration. Results obtained by Arup using a shaking table at Imperial College are shown in Fig. 2 which has two other results added. It can be seen that the fundamental component of lateral force increases with increasing platform amplitude but is insensitive to pavement lateral frequency. However, walkers were not asked to try to 'tune' their step intentionally to the platform's motion; instead they were asked to walk comfortably for the seven or eight paces required to pass over the platform. Figure 2 has three added lines which show the ratio of dynamic lateral force to static weight for a rigid mass when oscillated at 0.75 Hz (bottom line), 0.85 Hz (middle line) and 0.95 Hz (top line) when the amplitude of oscillation increases from 15 mm on the left-hand side to 35 mm on the right-hand side. This would apply if a pedestrian were modelled as a rigid mass whose centre of mass moved through an amplitude of 15 mm on a stationary pavement and increased linearly with increasing pavement amplitude to 35 mm when the pavement amplitude became 30 mm.

Fujino *et al.* [**3**] noticed that a person's head movement is typically twice that of their feet (laterally) at 1 Hz and ± 10 mm pavement movement, and so it is not surprising that pedestrians do not behave as rigid bodies. However, although they do not act as rigid masses, the lateral force that a person generates must be

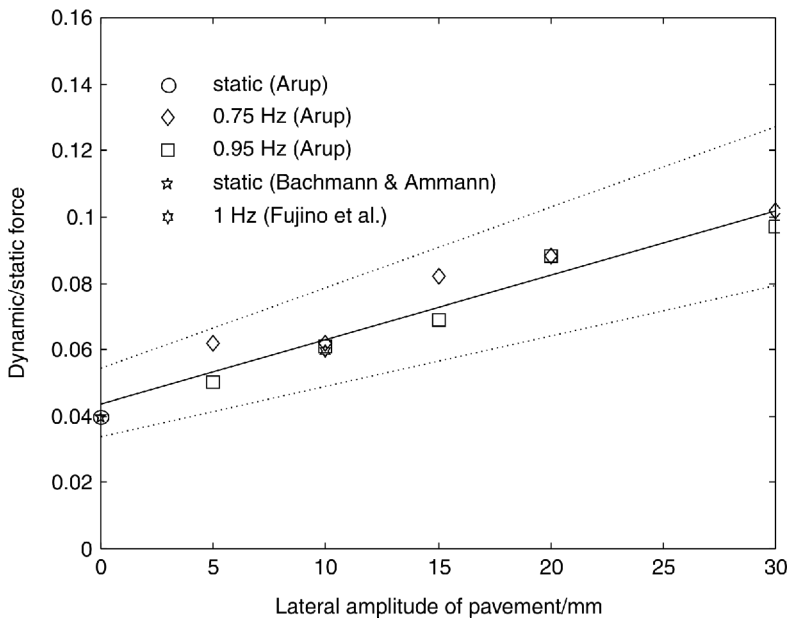

Fig. 2 Measured values of pedestrian lateral dynamic force/static weight as functions of pavement amplitude (after reference [**4**], Fig. 10) with data by Bachmann and Ammann [**1**] and Fujino *et al.* [**3**] added. The platform in these experiments was 7.3 m long and 0.6 m free width with a handrail along one side. The amplitude of the fundamental component of lateral force is plotted after dividing by the subject's weight. Arup's data are for two different frequencies of pavement oscillation: 0.75 and 0.95 Hz. It appears that subjects walked at a comfortable speed with a walking pace not intentionally 'tuned' to the pavement frequency. The data point from the paper by Fujino *et al.* shows an estimated force amplitude from observations of people walking on a bridge with a 1 Hz lateral mode at an amplitude of about 10 mm. The three added lines (drawn for comparison) are for moving a rigid mass at frequencies of 0.75 Hz (bottom line), 0.85 Hz (middle line) and 0.95 Hz (top line) for an amplitude of 15 mm (on the left) to 35 mm (on the right)

reacted against the inertia of their body, so that the sum of mass × acceleration for all their component parts must equal the lateral pavement force at all times. Therefore, if an average is calculated for each pedestrian, the results in Fig. 2 suggest that their centre of mass must be moving about ± 15 mm when walking comfortably on a stationary pavement, increasing linearly to about ± 35 mm when the pavement's lateral movement is ± 30 mm. The effect of frequency of pavement movement does not seem to have much effect, with measured data for 0.95 Hz suggesting a slightly lower ratio of dynamic to static force than 0.75 Hz. This is consistent with the natural flexibility of the human frame. Evidently, if the pavement were oscillating at a high frequency, the feet and legs would be expected to move, but the upper body would not follow so much and would move relatively less. Movement of the centre of mass of a pedestrian would then be significantly different from movement of the pavement.

To quantify this effect, a non-dimensional amplitude factor α is defined as the ratio of the movement of a person's centre of mass to the movement of the pavement. Since, from Fig. 2, it is deduced that a change in pavement amplitude from 0 to 30 mm causes a change in body movement from 15 to 35 mm, $\alpha = (35 - 15)/30 = 2/3$. These data suggest that α is approximately 2/3 at both 0.75 Hz and 0.95 Hz. Because of the complex dynamics of the human frame, it is possible that the effect of different frequencies in this range is small, as these data suggest.

Arup also studied the probability of synchronization for people using the walking platform at Imperial College and their results are shown in Fig. 3. This is the estimated probability that people will synchronize their footfall to the swaying frequency of the platform. The 'best-fit' straight line does not pass through the origin. It suggests that people synchronize with each other when there is no pavement motion but that the probability of synchronization increases as pavement amplitude increases. In calculations at the end of this paper, it will be assumed that the probability of synchronization, to be given by the symbol β, is 0.4 for platform amplitudes up to 10 mm.

In addition to these laboratory tests, Arup conducted a series of crowd tests on the Millennium Bridge. These concluded that pedestrian movement was strongly correlated with lateral movement of the bridge but not with vertical movement. This was attributed in part to the conclusion that pedestrians are 'less stable laterally than vertically, which leads to them being more sensitive to lateral vibration' (reference [**4**], p. 26). However, it was not concluded that vertical synchronization could not occur, and vibration control measures were added to the bridge in the expectation that vertical synchronization was a possibility.

In the following analysis, the assumptions that will be made about pedestrian loading are only appropriate for small-amplitude pavement movements (less than about 10 mm amplitude). For larger amplitudes, people's natural walking gait is modified as they begin to lose their balance and have to compensate by altering how they walk. The staggering movement of pedestrians trying to walk on a pavement which has large-amplitude lateral vibration (100 mm amplitude) has been studied by McRobie and Morgenthal [**6**] using a swinging platform. Pedestrian movement was followed by a motion capture system devised by Lasenby (see references [**7**] and [**8**]). The way that people walked on a platform moving with such a large amplitude varied from person to person. 'A common response was to spread the feet further apart and to walk at the same frequency as the pre-existing oscillations such that feet

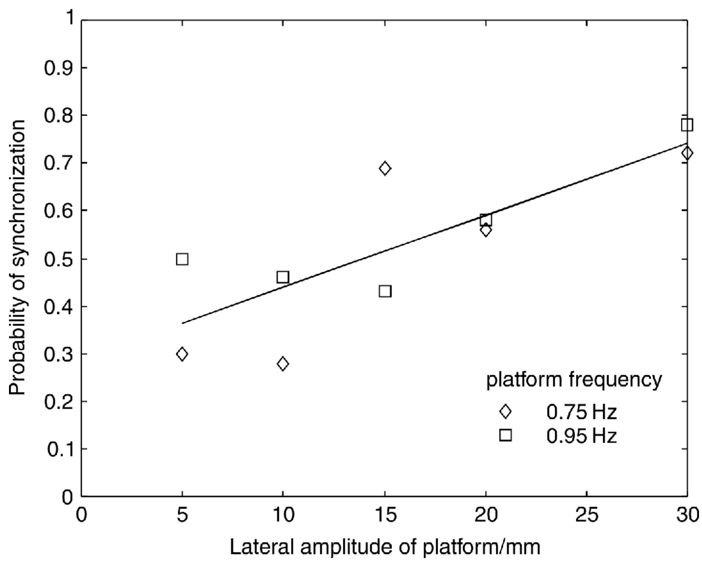

Fig. 3 Probability of synchronization estimated by Arup from moving-platform tests for the same two frequencies of platform lateral oscillation as in Fig. 2, 0.75 Hz and 0.95 Hz. (After reference [**4**], Fig. 1)

and deck maintained a constant phase relation.' However, 'Other walking patterns, some involving crossing of the feet, some involving walking in undulating lines were also observed' [**6**]. They also found that the lateral forces of the feet-apart gait are phase synchronized to the structure and approach 300 N amplitude per person, which these researchers pointed out is four times the Eurocode DLM1 value of 70 N for normal walking.

5 SMALL-AMPLITUDE PEDESTRIAN LOADING MODEL

The previous considerations lead to the notion that the force exerted by a walking pedestrian can be modelled (approximately) as two mass × acceleration terms. The first arises from the natural displacement of a person's centre of mass while walking on a stationary pavement, and the second from the additional displacement that occurs as a consequence of movement of the pavement. Let $x(t)$ be the natural movement of the centre of mass on a stationary pavement and $y(t)$ the movement of the pavement; then, if $f_0(t)$ is the force per person of mass m exerted on the pavement,

$$f_0(t) = m\ddot{x}(t) + m\alpha\ddot{y}(t - \Delta) \tag{1}$$

where Δ is a time lag to account for the fact that a person's centre of mass will generally not move in phase with movement of the pavement and α is a proportionality factor relating centre-of-mass movement to pavement movement. For lateral forces, the data in Fig. 2 suggest, for walking comfortably at about 0.85 Hz (1.7 paces/s) that a suitable value for α is 2/3 and that the amplitude of $x(t)$ would be about $|x| = 15$ mm. These are the values used to plot the middle line in Fig. 2.

The same equation (1) applies for modal quantities when there are many pedestrians provided that, at any point on the bridge, all of them are moving together. The modal force $f(t)$ now replaces the force per person $f_0(t)$, m becomes the modal mass of pedestrians (i.e. the distributed pedestrian mass normalized by the square of the bridge's displacement mode function), and x and y become modal displacements (both normalized by the bridge's displacement mode function).

However, it has been found experimentally that pedestrians do not always synchronize their steps (see Fig. 3), and only forces that are synchronized (i.e. correlated in time) cause bridge vibration, the uncorrelated forces cancelling each other out. Therefore we define two correlation coefficients: β to describe the correlation of the natural swaying movements that people make walking on a stationary platform, and γ to describe the correlation of the pedestrian movements that depend on platform movement. Introducing these factors into equation (1) gives

$$f(t) = m\beta\ddot{x}(t) + m\alpha\gamma\ddot{y}(t - \Delta) \tag{2}$$

For small pavement movements (up to 10 mm amplitude), there are currently insufficient experimental data to determine whether β and γ are different and it will be assumed that they are the same (and constant), so that

$$\beta = \gamma = \text{constant} \tag{3}$$

The upshot is that it will be assumed that the following equation applies to describe how the modal bridge excitation force arising from pedestrian motion depends on the modal accelerations of the people and the bridge, and the modal mass of pedestrians. It includes the two empirical factors α, which is the ratio of movement of a person's centre of mass to movement of the pavement, and β, which is a correlation factor for when individual people's movements synchronize. Thus, to a first approximation, the bridge loading to be expected from walking pedestrians will be modelled by the equation

$$f(t) = m\beta\ddot{x}(t) + m\alpha\beta\ddot{y}(t - \Delta) \tag{4}$$

The definitions of all terms are given in the notation section and the quantities in equation (4) can be interpreted either as modal quantities or as local quantities per unit length.

Note that the presence of pedestrians is assumed not to alter the bridge's modal properties which are satisfactorily described by small damping theory. Pedestrians act only as a forcing function for the bridge modes, generating a force defined by equation (4). This will now be used to compute bridge response.

6 ANALYSIS OF PEDESTRIAN–BRIDGE INTERACTION

The following calculation explores the interaction between pedestrians of effective modal mass m walking on a bridge with a vibration mode of (modal) mass M and stiffness K using the above (small-amplitude) force model. The interaction (modal) force which is transmitted from the pedestrians to the bridge, and vice versa, is f. This system is shown in Fig. 4 where $z(t)$ is

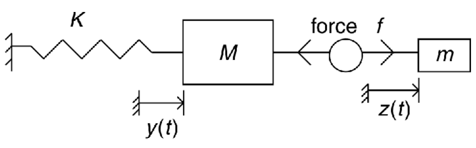

Fig. 4 Interaction between a bridge mode with modal mass M and stiffness K and pedestrians with modal mass m. The (modal) force transmitted between the pavement and pedestrians is f. The large open circle recognizes that there is a complex interaction between pedestrians and bridge recognized by the time delay Δ and the correction factors α and β in equation (7)

the (effective) modal displacement of the pedestrians' centre of mass and $y(t)$ measures the modal displacement of the bridge's pavement or walkway.

The equations of motion are

$$M\ddot{y}(t) + K\ddot{y}(t) = -f(t) \tag{5}$$

$$f(t) = m\ddot{z}(t) \tag{6}$$

where, using equation (4),

$$m\ddot{z}(t) = m\beta\ddot{x}(t) + m\alpha\beta\ddot{y}(t - \Delta) \tag{7}$$

so that

$$M\ddot{y}(t) + Ky(t) + m\alpha\beta\ddot{y}(t - \Delta) = -m\beta\ddot{x}(t) \tag{8}$$

where x and y are modal displacements, and α and β are defined above. If the modal natural frequency is $\omega_n = \sqrt{K/M}$ and if damping is included with a damping ratio ς, then equation (8) may be written

$$\ddot{y}(t) + \frac{m}{M}\alpha\beta\ddot{y}(t - \Delta) + 2\varsigma\omega_n\dot{y}(t) + \omega_n^2 y(t) = -\frac{m}{M}\beta\ddot{x}(t) \tag{9}$$

If the natural movement of a pedestrian (on a stationary pavement), $x(t)$, is known, then equation (9) can be solved for the modal bridge displacement $y(t)$). Of course, only small-amplitude movement of the pavement is considered so that pedestrians walk unimpeded by motion of the bridge, without any pronounced change in gait. The time lag Δ allows for the possible delay in body following feet. This is likely to be a greater factor in lateral vibration than vertical vibration because the body can sway slightly laterally and then the pedestrian may easily have a body movement which is out of phase with the motion of their feet. For brevity in the following analysis, the mass ratio parameter m_r is now defined as

$$m_r = \frac{\alpha\beta m}{M} \tag{10}$$

and a frequency ratio as

$$\Omega = \frac{\omega}{\omega_n} \tag{11}$$

where m/M is the ratio of pedestrian mass to bridge mass (either modal or per unit length when the parameters are constant along the bridge), α is the ratio of movement of a person's centre-of-mass to movement of their feet, β is the proportion of pedestrians whose movement has synchronised with pavement movement and Ω is the ratio of frequency of excitation to bridge natural frequency.

6.1 Frequency analysis

By taking Fourier transforms of both sides of equation (9), and putting

$$Y(\mathrm{i}\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} y(t)\exp(-\mathrm{i}\omega t)\,\mathrm{d}t \tag{12}$$

and similarly for $Z(\mathrm{i}\omega)$,

$$Y(\mathrm{i}\omega)\left[-\omega^2 + 2\varsigma\omega_n\mathrm{i}\omega + \omega_n^2 - m_r\omega^2\exp(-\mathrm{i}\phi)\right] + X(\mathrm{i}\omega)\frac{m_r}{\alpha}(-\omega^2) = 0 \tag{13}$$

where the phase angle ϕ is given by

$$\phi = \omega\Delta \tag{14}$$

6.2 Singular solution for $X(\mathrm{i}\omega) = 0$

Firstly, it is assumed that $x(t) = 0$ so that $X(\mathrm{i}\omega) = 0$. This means that, on a stationary pavement, a pedestrian can walk without introducing any dynamic force. The pedestrian's weight glides forwards in the direction of walking without any up-and-down or side-to-side movement of their centre of mass, and no force is exerted on the pavement (except static weight). If there is no time delay so that $\Delta = 0$ in equation (5), and therefore $\phi = 0$ in equation (9), the only solution is $Y = 0$ and no vibration occurs. However, if some time delay can occur, there is the possibility of a non-zero solution for Y.

This non-zero solution can be found as follows. After separating the real and imaginary parts in equation (13), this becomes

$$Y(\mathrm{i}\omega)[(\omega_n^2 - \omega^2 - m_r\omega^2\cos\phi) + \mathrm{i}\omega(2\varsigma\omega_n + m_r\omega\sin\phi)] \tag{15}$$

which has a non-zero solution for Y (actually an indeterminate solution) if the real and imaginary parts within the square brackets are both zero, so that

$$\omega_n^2 - \omega^2 - m_r\omega^2\cos\phi = 0 \tag{16}$$

$$2\varsigma\omega_n + m_r\omega\sin\phi = 0 \tag{17}$$

which, on eliminating ϕ, lead to the following expression for the damping ratio ς:

$$4\varsigma^2 = 2 + \Omega^2(m_r^2 - 1) - \frac{1}{\Omega^2} \tag{18}$$

For this solution, the phase angle ϕ can be calculated from either equation (16) or equation (17) to give

$$\phi = \sin^{-1}\left(\frac{-2\varsigma}{\Omega m_r}\right) \tag{19}$$

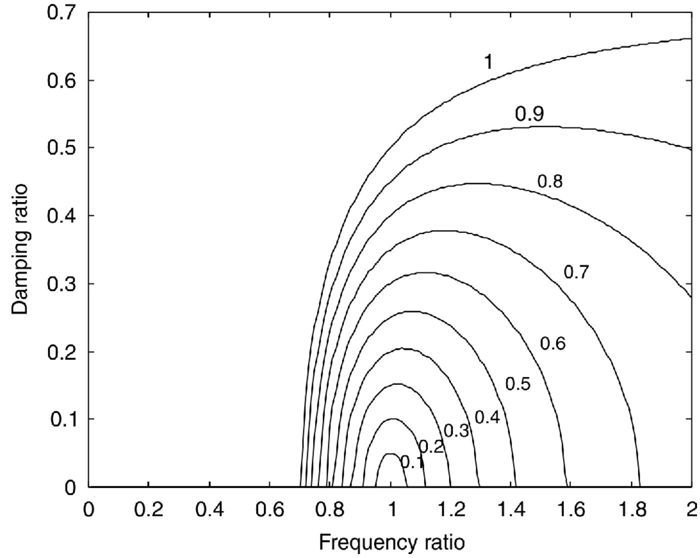

Fig. 5 Damping ratio required for stability as a function of frequency ratio for different mass ratios—from 0.1 to 1. The unstable region lies below the curve in each case

Results from equations (18) and (19) are plotted in Figs 5 and 6.

It can be seen that, for small pedestrian-to-bridge mass ratio m_r, only a small amount of damping is needed to ensure stability and there is only a narrow band of frequencies, close to the natural frequency, at which self-excitation can occur. However, for higher mass ratios, the required damping to maintain stability increases and the range of frequencies over which self-excitation can occur increases greatly. The phase angle by which pedestrian body movement leads bridge movement for this limiting motion is shown in Fig. 6. It is close to 90° when the frequency ratio is close to unity, i.e. when the excitation frequency of pedestrian loading is close to the natural frequency of pavement motion.

6.3 Critical damping ratio for stability

Figure 5 shows the minimum damping needed to give stability for any chosen mass ratio m_r. Consider, for

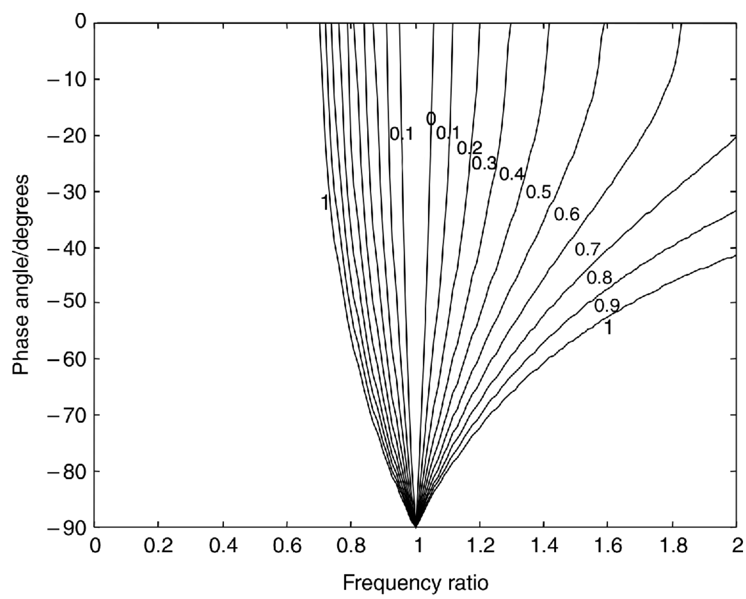

Fig. 6 Phase angle between pedestrian motion and pavement motion (at the stability limit) plotted against frequency ratio for different mass ratios m_r. Since ϕ is defined as positive when pedestrian motion lags pavement motion, in the graph pedestrian motion leads pavement motion by 90° when the frequency ratio is unity

example, the curve for $m_r = 0.5$. The system will be prone to self-excite at any frequency in the range 0.8–1.4 approximately. In theory, any frequency ω is possible (as the only input from the pedestrian is to influence the phase angle ϕ by responding to pavement movement at whatever frequency this occurs). Therefore, unless the damping ratio exceeds the maximum shown on this curve, about 0.27, self-excitation will occur at a frequency ratio close to unity.

By calculating the maxima of the curves drawn in Fig. 5, the critical damping ratio required for stability (whatever the frequency) and the frequency at which this instability first occurs if it does occur can be found. After differentiating equation (18) with respect to ω using equation (11), with m_r constant, maxima are found to occur at a critical frequency ω_c where

$$\left(\frac{\omega_c}{\omega_n}\right)^4 = \frac{1}{1 - m_r^2} \tag{20}$$

and the value of the critical damping ratio ς_c is

$$\varsigma_c^2 = \tfrac{1}{2}\left(1 - \sqrt{1 - m_r^2}\right) \tag{21}$$

When the mass ratio m_r is small, the right-hand side of equation (21) can be expanded by the binomial theorem, to give

$$2\varsigma_c \approx m_r, \qquad m_r \ll 1 \tag{22}$$

Similarly, from equation (20), the corresponding critical frequency ω_c is

$$\frac{\omega_c}{\omega_n} \approx 1 + \frac{m_r^2}{4} \tag{23}$$

and, from equation (19), the critical phase angle ϕ_c is

$$\phi_c \approx -\frac{\pi}{2} + \frac{m_r}{2} \tag{24}$$

when higher-order terms are neglected.

The damping ratio [equations (21) and (22)] is the damping for which an oscillatory motion can, in theory, occur at (any) constant amplitude. For greater damping, vibration if started, subsides. For lesser damping, vibration builds up spontaneously. These results are plotted in Figs 7 and 8.

The curve in Fig. 5 for $m_r = 1$ becomes asymptotic to the damping ratio $\varsigma = 1/\sqrt{2} = 0.7071$, and curves for higher mass ratios than unity are monotonically increasing, with no maxima. For such high pedestrian loading ratios, according to this response model, a stable solution is not possible, whatever the damping ratio.

6.4 Forced vibration solution

Now the forced vibration solution when $X(\mathrm{i}\omega) \neq 0$ is considered. Suppose that the natural movement of a pedestrian's centre of mass, when walking steadily on a stationary pavement, is

$$x(t) = X \exp(\mathrm{i}\omega t) \tag{25}$$

and that the resulting pavement response is

$$y(t) = Y \exp(\mathrm{i}\omega t) \tag{26}$$

On substituting equations (25) and (26) into equation (9), or directly from equation (13), the result is that the

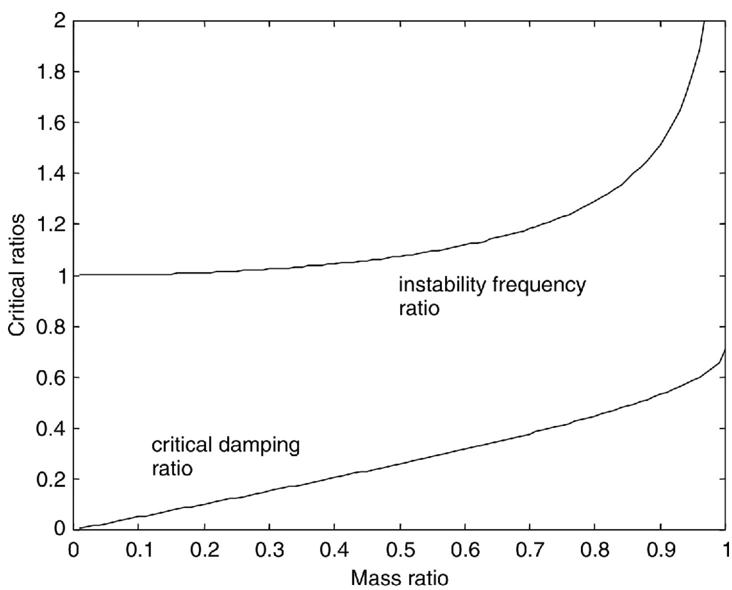

Fig. 7 Critical damping ratio required for stability (lower curve) and frequency ratio at which instability first occurs (upper curve) as functions of mass ratio

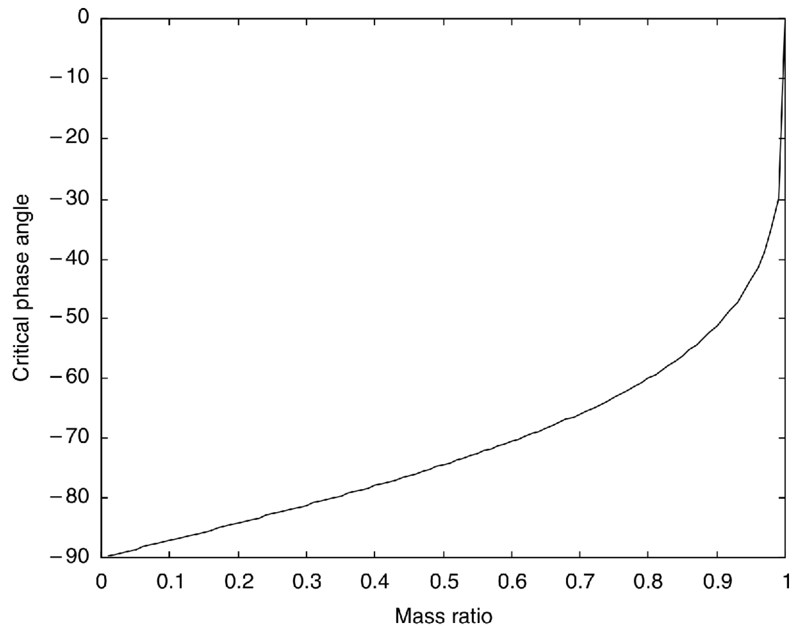

Fig. 8 Critical phase angle between relative movement of body and feet at the stability limit, plotted as a function of mass ratio

amplitude of the pavement response is given by

$$\left|\frac{\alpha Y}{X}\right| = m_{\mathrm{r}}\Omega^2\left\{[1-\Omega^2(1+m_{\mathrm{r}}\cos\phi)]^2 + \left[2\varsigma\Omega+\Omega^2 m_{\mathrm{r}}\sin\phi\right]^2\right\}^{-1/2} \tag{27}$$

and the pavement response lags behind the pedestrian excitation by angle θ where

$$\theta = \tan^{-1}\left[\frac{2\varsigma\Omega+\Omega^2 m_{\mathrm{r}}\sin\phi}{1-\Omega^2(1+m_{\mathrm{r}}\cos\phi)}\right] \tag{28}$$

For equations (27) and (28) to describe the total motion that is occurring, any transient motions must have decayed. This will only happen if the damping ratio of the bridge exceeds the critical value given by equations (21) and (22) above.

The amplitude of response given by equation (27) depends on the phase angle ϕ by which pedestrian body movement lags movement of their feet. Figure 9 shows four curves for the case when $m_{\mathrm{r}} = 0.1, \varsigma = 0.1$, for $\phi = 0, -\pi/2, -\pi$ and $\pi/2$ in which $|\alpha Y/X|$ is plotted against Ω. Since the worst-case scenario is sought, the curve which gives the highest response is needed. Rather than plot all possible curves for all values of ϕ, the upper envelope of these curves can be plotted, as shown in Fig. 10. The equation for this envelope can be calculated from equation (27) by differentiating with respect to ϕ to seek a maximum. This occurs when

$$\phi = \tan^{-1}\left(-\frac{2\varsigma\Omega}{1-\Omega^2}\right) \tag{29}$$

and is given by

$$\left|\frac{\alpha Y}{X}\right|_{\max} = m_{\mathrm{r}}\Omega^2\left[\left(1-\Omega^2\right)^2+4\varsigma^2\Omega^2+m_{\mathrm{r}}^2\Omega^4 - 2m_{\mathrm{r}}\Omega^2\sqrt{\left(1-\Omega^2\right)^2+4\varsigma^2\Omega^2}\right]^{-1/2} \tag{30}$$

Equation (30) is used to compute the results in Figs 11 and 12. These results assume that a stable solution is possible, which is only the case if $\varsigma > \varsigma_{\mathrm{c}}$, where ς_{c} is the critical damping ratio required for stability and is given by equation (21) or, for $m_{\mathrm{r}} \ll 1$ by equation (22). Figure 11 has mass ratio $m_{\mathrm{r}} = 0.1$ and Fig. 12 a larger ratio $m_{\mathrm{r}} = 0.3$. For both graphs (Figs 11 and 12), the damping is chosen so that each graph shows curves for $\varsigma/\varsigma_{\mathrm{c}} = 1.1, 1.3, 1.5, 2, 3$ and 5. Note that the critical damping c_{c} depends on m_{r} according to equations (21) and (22).

6.5 Effective damping ratio

From Figs 11 and 12, the forced response is clearly very similar in its appearance to the forced resonance of a single-degree-of-freedom system. Provided that the bridge's damping is only slightly higher than the critical

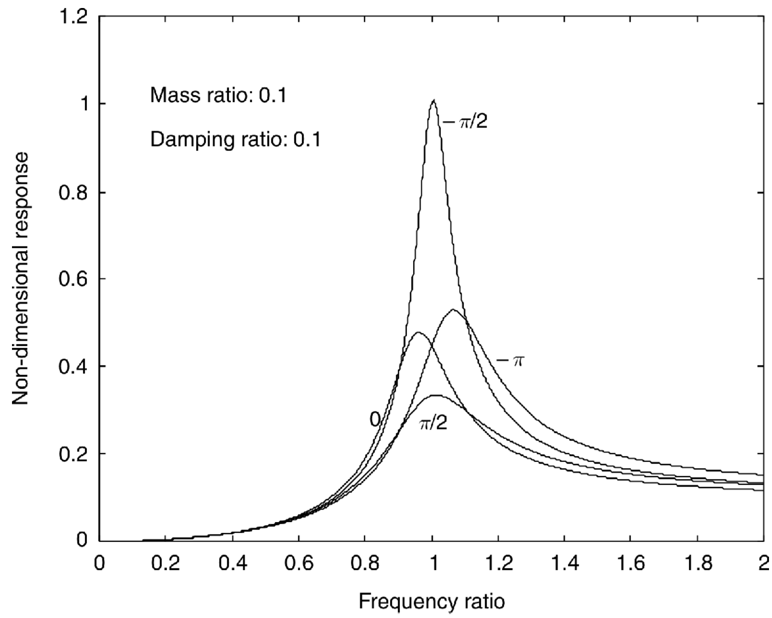

Fig. 9 Forced response of bridge for $m_r = 0.1, \varsigma = 0.1$ and four different phase angles ϕ. The ordinate is the non-dimensional response $|\alpha Y/X|$ defined by equation (27)

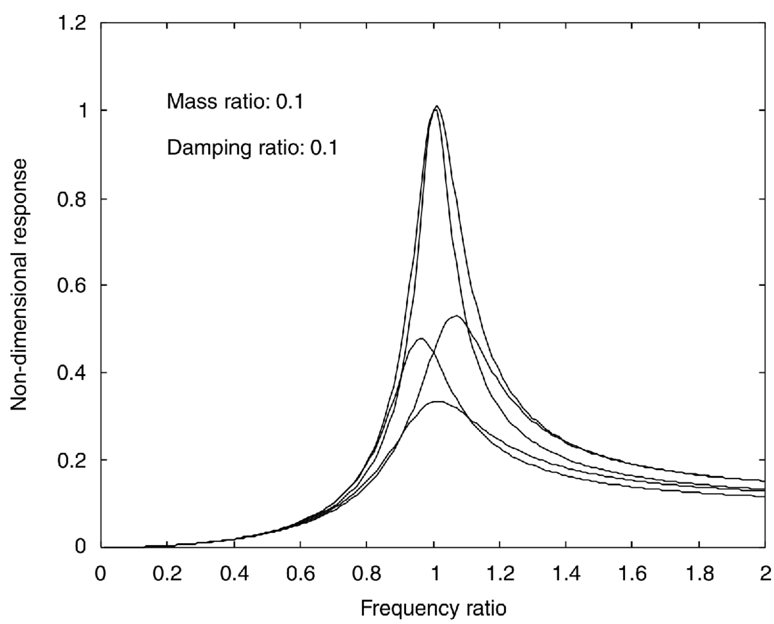

Fig. 10 The same as Fig. 9 but showing the upper envelope of all the curves

damping for stability, vibration occurs at a frequency close to the natural frequency of the relevant mode.

Consider a single degree-of-freedom system with displacement response $Y \exp(\mathrm{i}\omega t)$ when subjected to a harmonic force $F \exp(\mathrm{i}\omega t)$. The amplitude of its steady state response is given by

$$\left|\frac{kY}{F}\right| = \left[\left(1 - \Omega^2\right)^2 + (2\varsigma_{\mathrm{eff}}\Omega)^2\right]^{-1/2} \tag{31}$$

where k is the stiffness, Ω the frequency ratio [equation (11)] and ς_{eff} is the effective damping ratio. When the force $F(t)$ comes from the inertial loading of a mass βm moving harmonically through distance X then

$$F(t) = -\beta m\omega^2 X \exp(\mathrm{i}\omega t) = F \exp(\mathrm{i}\omega t) \tag{32}$$

After substituting in equation (31) for F and sorting terms gives

$$\left|\frac{\alpha Y}{X}\right| = m_r\Omega^2\left[\left(1 - \Omega^2\right)^2 + (2\varsigma_{\mathrm{eff}}\Omega)^2\right]^{-1/2} \tag{33}$$

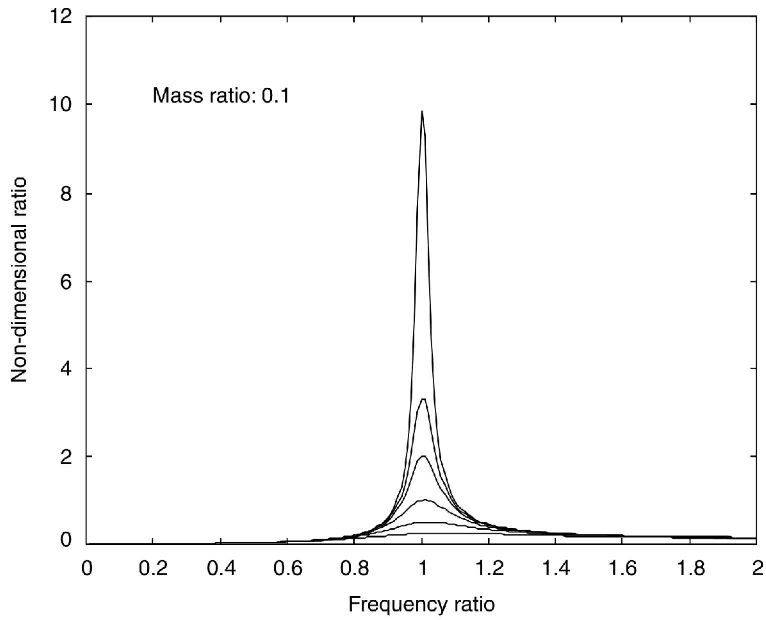

Fig. 11 Forced response of bridge for $m_r = 0.1$ and six different damping ratios given by $\varsigma/\varsigma_c = 1.1, 1.3, 1.5, 2, 3$ and 5. The ordinate is the non-dimensional response $|\alpha Y/X|_{max}$ defined by equation (30)

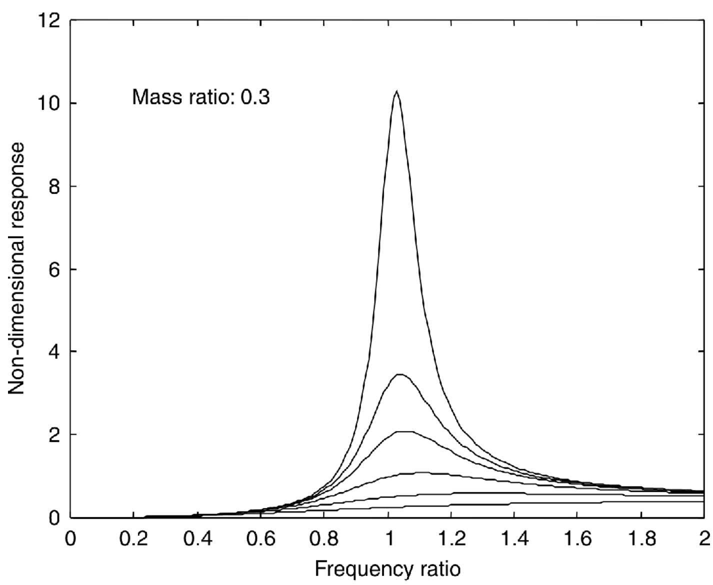

Fig. 12 The same as Fig. 11 except that the mass ratio $m_r = 0.3$ instead of 0.1

At resonance ($\Omega = 1$), then

$$\left|\frac{\alpha Y}{X}\right|_{\Omega=1} = \frac{m_r}{2\varsigma_{eff}} \tag{34}$$

Compare the corresponding result obtained from equation (30) by putting $\Omega = 1$:

$$\left|\frac{\alpha Y}{X}\right|_{max,\,\Omega=1} = \frac{m_r}{2\varsigma - m_r} \tag{35}$$

The peak height of the response curves calculated by equations (30) and (33) will therefore be the same if

$$\varsigma_{eff} = \varsigma - \frac{m_r}{2} \tag{36}$$

so that the effective damping ratio can be calculated by subtracting $m_r/2$ from the actual (structural) damping ratio of the bridge mode concerned. Using the definition of ς_c in equation (22), equation (36) may alteratively be written as

$$\varsigma_{eff} = \varsigma - \varsigma_c \tag{37}$$

provided that the damping is small and $m_r = \alpha\beta m/M$ is small.

7 PEDESTRIAN SCRUTON NUMBER

McRobie and Morgenthal [9] have pointed out the analogy between wind excitation and people excitation. The tendency for vortex shedding to excite structural oscillations is measured by the non-dimensional Scruton number S_c which is a product of damping and the ratio of representative structural and fluid masses. The usual definition is

$$S_c = \frac{4\pi\varsigma M}{\rho b^2} \tag{38}$$

where ς is the damping ratio of the relevant mode, ρ is the air density and, for a cylindrical structure of diameter b, M is the mass per unit length of the structure. Large Scruton numbers are preferable. McRobie and Morgenthal suggested that the same approach should be taken for pedestrian-excited vibration, distinguishing between vertical and lateral vibration to allow for the different human responses to vertical and lateral pavement movement. The definition of pedestrian Scruton number S_{c_p} is arbitrary but, for the purpose of this paper, by comparison with equation (38), it is defined as

$$S_{c_p} = \frac{2\varsigma M}{m} \tag{39}$$

where

S_{c_p} = pedestrian Scruton number

ς = modal damping ratio

M = modal mass or, for a uniform deck, bridge mass per unit length

m = modal mass of pedestrians or, for a uniform bridge deck with evenly spaced pedestrians, pedestrian mass per unit length

For this definition, in order to exceed the minimum damping given by equation (22), it is necessary that

$$S_{c_p} > \alpha\beta \tag{40}$$

This analysis uses the model in equation (4). As before, α is the ratio of movement of a person's centre of mass to movement of the pavement, which from the measured results above is typically 2/3 for lateral vibration in the frequency range 0.75–0.95 Hz, and β is the correlation factor for individual people to synchronize with pavement movement, which is typically 0.4 for lateral pavement amplitudes less than 10 mm.

Typical data have been assembled from the sources available (which are somewhat meagre and generally incomplete) and is reproduced in Fig. 13 (for lateral vibration) and Fig. 14 (for vertical vibration). Each figure has two horizontal lines showing S_{c_p} calculated from equation (40) for the case when $\alpha = 2/3$ and $\beta = 0.4$ (lower limit) and when $\alpha = 1$ and $\beta = 1$ (upper limit). Depending on the values of these empirical factors, the horizontal lines show the minimum pedestrian Scruton number required for stability.

It can be seen that for typical modes of the London Millennium Bridge the pedestrian Scruton numbers were initially very low, less than the lower limit. After modification to increase artificially the bridge's damping, the corresponding S_{c_p} are much higher, well above the upper limit drawn and also well above an alternative limit (42) suggested by Arup (see below). Of course at present there is, as already mentioned, only limited experimental data and the 'best' values to use for α and β remain to be established. Although β (the correlation factor for when people synchronize with pavement movement) cannot exceed unity, it is possible that α (the ratio of body movement to pavement movement) may be greater than unity and then the upper limit drawn for $\alpha = \beta = 1$ would be higher. Interestingly, a recent study of lateral vibration by Roberts [10], published while this paper was in press, suggests that it is plausible that the limit of stability will occur when $\alpha = 1$. This (unproven) assumption leads to an expression for the maximum number of pedestrians permissible on a bridge of given length and dynamic properties if instability is to be prevented. Although apparently different, when expressed as a non-dimensional Scruton number, Roberts' stability criterion can be shown to reduce to $S_{c_p} = 1$, the upper limit in Fig. 13. At the time of completion of this paper, various new studies of the dynamics of pedestrian bridges are taking place which may provide much needed further data but, until they do, uncertainty remains.

For lateral vibration (Fig. 13), the estimated pedestrian Scruton numbers for both the bridge in Japan studied by Fujino *et al.* and the Auckland Harbour Bridge lie below the lower limit from equation (40). In the case of vertical vibration (Fig. 14), additional data were given by McRobie and Morgenthal [9] for some other bridges that have caused concern, all of which fall below the limit. The data for Auckland Harbour Bridge are interesting because they fall between the upper and lower limits from equation (40). This relates to data measured during the course of a marathon race in 1992 when a large number of runners crossed one of the two-lane roadways. It is recorded that vertical amplitudes of up to 3 mm were experienced in a frequency range 2.6–3 Hz, which is noticeable by runners. From this it may be concluded that vertical bridge oscillation of serious amplitudes could be excited by the natural synchronization of a large enough crowd of runners (as distinct from a marching army in the traditional sense). That is why the decision was taken to increase artificially the

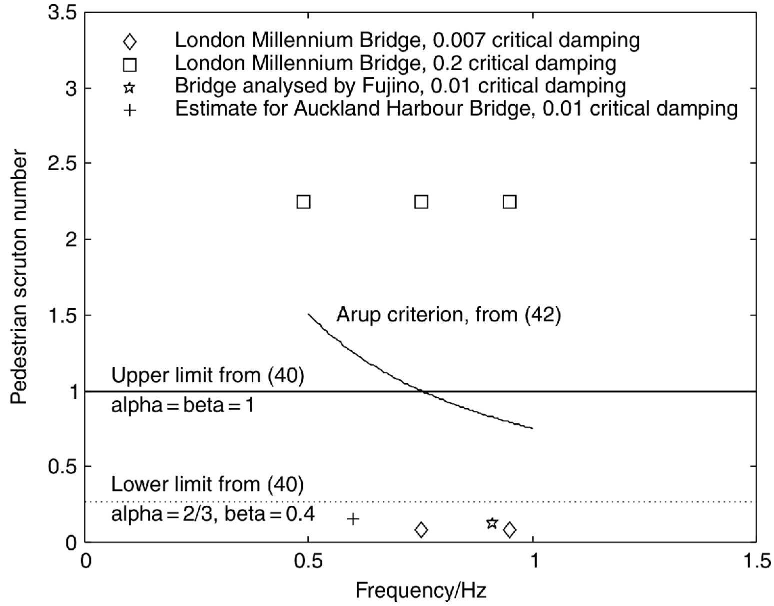

Fig. 13 Some collected data on the pedestrian Scruton number for lateral modes. The critical value from equation (42) is plotted for comparison

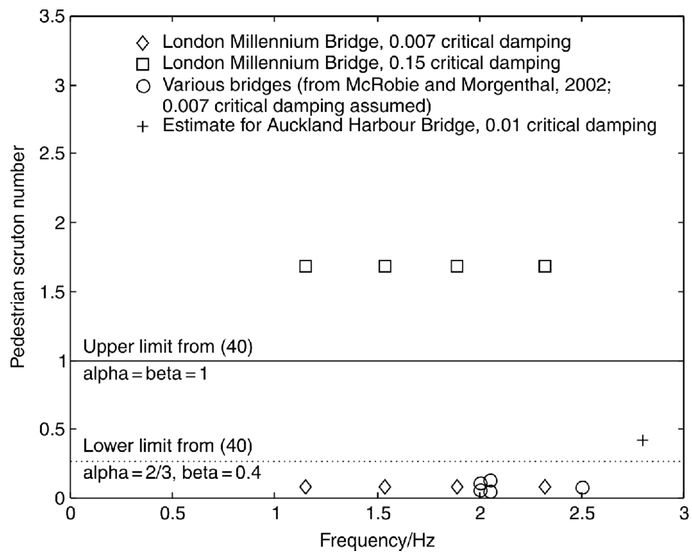

Fig. 14 Some collected data on the pedestrian Scruton number for vertical modes

damping of vertical as well as lateral modes for the London Millennium Bridge.

8 ARUP'S ANALYSIS

As a result of a series of crowd tests, and an energy analysis of vibrational power flow, Arup concluded that the correlated lateral force per person is related to the local velocity by an approximately linear relationship which was found to hold for lateral frequencies in the range 0.5–1 Hz (pacing frequency 1–2 Hz). It is interesting that, within this frequency range, the results again appear to be insensitive to frequency. If k is defined as the slope of a graph of the amplitude of average lateral force per person plotted against the amplitude of pavement lateral velocity, Arup found that $k \approx 300\,\text{N s/m}$ for a bridge mode in the frequency range 0.5–1 Hz (see reference [**4**], p. 27).

By assuming that each person generates a velocity-dependent force which acts as negative damping, and making a modal calculation, they also concluded that vibrational energy in the mode would not increase if

$$\varsigma > \frac{Nk}{8\pi f_n M} \tag{41}$$

where ς is the modal damping ratio, f_n is the natural frequency and M is the modal mass [4, equation (9)]. For this condition, the positive modal damping exceeds the negative damping generated by pedestrian movement. On substituting equation (41) into equation (39) to calculate the required pedestrian Scruton number, it is found that this limiting condition can be expressed as

$$S_{c_p} > \frac{k}{2\pi f_n m_0} \tag{42}$$

where m_0 is the mass per person for whom $k = 300\,\text{N s/m}$ in the frequency range 0.5–1.0 Hz. This result has been added to Fig. 13 for the case when $m_0 = 63.5\,\text{kg}$. Equation (42) crosses the upper limit from equation (40) at a frequency of 0.75 Hz approximately.

If S_{c_p} is less than the limit in equation (42), vibration self-excites as people begin by walking normally and then progressively fall into synchronism with the pavement motion until this builds up to a level at which steady walking becomes impossible and a staggering movement takes over.

9 CONCLUSIONS

The analysis in this paper turns on assuming that, in the process of synchronization, the time lag that people take to respond to bridge movement (represented by Δ) naturally adjusts itself to have the greatest effect, subject to a correction factor to allow for the fact that only a proportion β of all pedestrians make this synchronization. By selecting the value of Δ to give the greatest response, it is concluded that bridge vibration will become unstable when the live load, represented by people of mass m per unit length, is too great a proportion of the bridge mass M per unit length.

The permissible m/M ratio depends on the amount of damping present in the appropriate vibrational modes that will be excited by the pacing rate of pedestrians because problems only arise when this excitation frequency is close to a natural frequency of a lightly damped vibrational mode. Specifically, according to the analysis, it is necessary for stability that, by combining equations (22) and (10),

$$2\varsigma > \frac{\alpha\beta m}{M} \tag{43}$$

where ς is the damping ratio in the mode and α and β are experimentally determined factors. From the data so far available, it appears satisfactory to assume that $\alpha = 2/3$ relates movement of a person's centre of mass to movement of their feet (from the slope in Fig. 2) and $\beta = 0.4$ for bridge amplitudes up to about 10 mm (from Fig. 3). However, these numbers derive from a limited number of experiments on lateral vibration and can only be regarded as provisional for lateral vibration and a first indication of possible numbers for vertical vibration for which measurements have not yet been made.

The dependence of a damping stability criterion on a mass ratio is consistent with the experience of vortex-excited oscillations in aeroelasticity. Application of the pedestrian Scruton number defined by equation (39) allows the criterion (43) to be expressed alternatively by equation (40). The results plotted in Figs 13 and 14 show that troublesome bridges all have pedestrian Scruton numbers that fall below the limit given by equation (40) with $\alpha = 2/3$ and $\beta = 0.4$. Similarly, the London Millennium Bridge after modification [11–13] lies well above the upper limit of the pedestrian Scruton number drawn by assuming that the factors α and β are both unity. This is a worst-case assumption for the correlation factor β and may be a pessimistic assumption for the value of the amplitude ratio α.

The collection of more experimental data is needed to verify these conclusions but, so far as the author knows, no unstable bridges whose pedestrian Scruton numbers lie above the upper limit shown in Figs 13 and 14 have yet been found.

ACKNOWLEDGEMENTS

The author is grateful to many people who have been involved, directly or indirectly, in this study including particularly Pat Dallard, Michael Willford, Roger Ridsdill Smith and the late Tony Fitzpatrick of Arup, David Malam of Atkins, Shayne Gooch at the University of Canterbury, New Zealand, and colleagues in Cambridge, Allan McRobie and Joan Lasenby. All these and many more with whom the author has not worked personally have contributed greatly to learning about how pedestrians and bridges interact. Research on this problem has built on the seminal work of Hugo Bachmann in Switzerland and Yozo Fujino in Japan whom, it is believed, were the first to identify and write about the issues discussed in this paper.

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