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Whirling Speeds of Flexibly-Mounted Rotors

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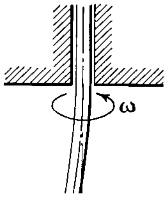
A rigid rotor mounted on a flexible shaft always has one whirling speed at which a small unbalance can produce very large deflections. Depending on the moments of inertia of the rotor, there may also be a second whirling speed. The theoretical calculation of these critical speeds is explained, and a convenient non-dimensional diagram presented which illustrates the dependence of the whirling speeds on the system parameters. An important practical aspect is that the gyroscopic moments involved may not only cause there to be a second whirling speed, but also affect the first speed. This may be appreciably lower than is indicated by a calculation which neglects the gyroscopics of the problem.

A ROTOR will not run smoothly if its rotational speed is near a critical whirling speed. For many applications the rotating system may be modelled by a rigid rotor mounted on a flexible massless shaft. In this case the system will always have one whirling speed. Depending on the system parameters, there may also be a second whirling speed. Usually the first depends principally on the mass and lateral stiffness of the shaft. However it is shown that this result is not always reliable, and that the gyroscopic moments of the spinning rotor given to illustrate the detailed calculation in a particular case.

WHIRLING MOTION OF A ROTOR

A typical flexibly mounted rotor is shown in Fig. 1. A rotationally symmetrical, rigid rotor is hanging on a flexible shaft from a single fixed bearing. The rotor is assumed to be perfectly balanced and the shaft straight, axis in this deformed shape. In theory the magnitude of δ is now indeterminate (although the ratio δ/θ is constant at each whirling speed). However the effect of a small unbalance would be to make δ very large, and limited only by the inherent damping and non-linearity of the system.



The dynamic forces acting on the rotor during a small amplitude whirling motion are shown in Fig. 3. If the rotational angular velocity (about the vertical axis) is ω, there is the centrifugal force $M\omega^2\delta$ and a gyroscopic torque about the centre of mass which is $(I_D-I_P)\omega^2\theta$, where M is the rotor mass and I_D and I_P are the diametral and polar moments of inertia about axes through the centre of mass. The relations between the deflection δ and rotation θ and this force and torque may now be expressed in terms of the influence coefficients α_{11}, α_{22} and $\alpha_{12}=\alpha_{21}$ for the elastic mounting.

$$\delta=\alpha_{11}\,[M\omega^2\delta]+\alpha_{12}\,[(I_D-I_P)\,\omega^2\theta] \quad . \quad . \quad (1)$$

$$0=\alpha_{21}\,[M\omega^2\delta]+\alpha_{22}\,[(I_D-I_P)\omega^2\theta] \quad . \quad . \quad (2)$$

δ/θ may be eliminated between these two equations to find the following frequency equation for the whirling speeds.

$$(\alpha_{11}\alpha_{22}-\alpha_{12}^2)\,(I_D-I_P)\,M\omega^4-[\alpha_{11}M+\alpha_{22}(I_D-I_P)\,]\omega^2+1=0$$

By putting

$$\Omega^2=\alpha_{11}\,M\omega^2$$

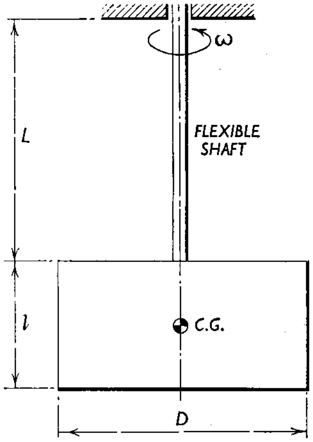


Fig. 1—Example of a flexibly mounted rigid rotor

may act so as to increase or decrease the first whirling speed by appreciable amounts. Furthermore, for rotors which are designed to operate above their first whirling speed, the calculation of the second, higher critical speed is extremely important. For instance, modern high-speed centrifuges are often provided with a very flexible shaft which allows the rotor to choose its own axis of rotation without causing high bearing loads. In this case the centrifuge rotor invariably operates above its first whirling speed, and it is then important to know whether the running speed approaches the second whirling speed.

The calculation of the critical speeds for whirling is outlined below for the general case of a symmetrical rotor on a flexible shaft. The dependency of the whirling speeds on the system parameters is conveniently shown on a non-dimensional diagram which plots the critical speeds against the difference between the diametral and polar moments of inertia of the rotor. Finally an example is

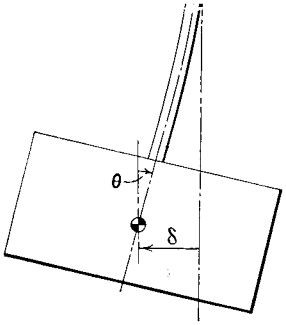


Fig. 2—Configuration during a possible whirl motion

so that, when there is no rotation, the rotor centre of mass lies directly below the bearing and the axis of symmetry is vertical. In general, when the rotational speed is increased the rotor will turn about its axis of symmetry and will remain in equilibrium without any displacement of the mass centre or any angular tilting. However at the whirling speeds the rotor can remain in equilibrium in some disturbed position, as shown in Fig. 2. The displacement δ and the angular deflection θ then remain constant, and the rotor now spins about the vertical

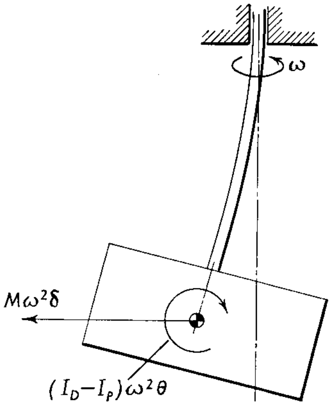


Fig. 3—Dynamic forces during whirling motion

$$\gamma=\frac{(I_D-I_P)\,\alpha_{22}}{M\alpha_{11}}$$

$$\beta=\frac{\alpha_{11}\,\alpha_{22}}{\alpha_{11}\,\alpha_{22}-\alpha_{12}{}^2}$$

this may be rewritten in the form

$$\frac{\gamma}{\beta}\Omega^4-(1+\gamma)\,\Omega^2+1=0 \quad . \quad . \quad . \quad . \quad (3)$$

which is a quadratic equation for Ω^2. Clearly the (non-dimensional) speed Ω must be a real, positive number, and so Ω^2 must be real, positive. From the theory of quadratic equations the solution Ω^2 will always be real if

$$(1+\gamma)^2>\frac{4\gamma}{\beta} \quad . \quad . \quad . \quad (4)$$

Now it is a property of the statics of elastic media† that $\alpha_{11}\,\alpha_{22}>\alpha_{12}{}^2$. Hence β is always a positive number greater than unity and so the inequality (4) is always satisfied. The solutions of Equation (3) are consequently always real. They will both be positive if

$$\gamma=\frac{(I_D-I_P)\,\alpha_{22}}{M\,\alpha_{11}}>0$$

Since the influence coefficients are always positive numbers, this means that there will only be two positive solutions, and therefore two whirling speeds, if $I_D-I_P)>0$ or, in words, if the diametral moment of inertia is greater than the polar moment of inertia. If $(I_D-I_P)<0$ or the diametral moment of inertia is less than the polar moment of inertia, then there will only be one positive value for Ω^2, and consequently only one whirling speed.

Whirling Speed Diagram

The solutions of Equation (3) for the whirling speeds may be conveniently represented on a diagram of (non-dimensional)

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† This condition follows from the requirement that the energy of deformation must be positive. See, for instance, Hildebrand *Methods of Applied Mathematics*, Prentice-Hall, Inc., page 50, 1952.

speed Ω^2 versus inertia difference γ, as shown in Fig. 4. In this diagram the region to the right of the origin has $I_D > I_P$, and in this region there are therefore two separate curves corresponding to the two different whirling speeds. The region to the left of the origin has $I_D < I_P$ and in this case there is only one curve for the one whirling speed.

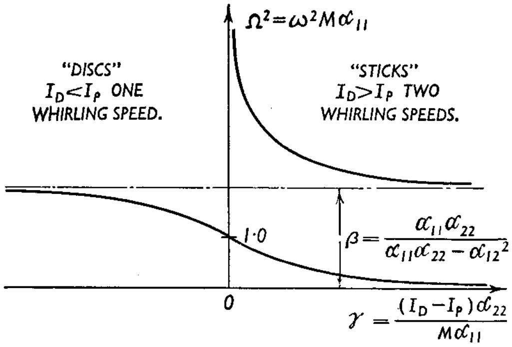
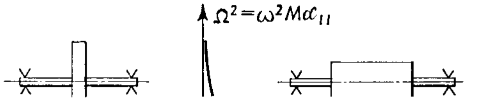


Fig. 4—Critical speeds

Qualitatively it may therefore be said that if a rotor is shaped like a stick, it will have $I_D > I_P$ and therefore two whirling speeds. If it is shaped like a thin disc it has $I_D < I_P$ and then there is only one whirling speed. For a solid cylindrical rotor, of diameter D and axial length l, the rotor is a "stick" for $l > 0{\cdot}866D$ and a "disc" for $l < 0{\cdot}866D$. In most practical cases it turns out that the rotor is a "stick" and consequently has two whirling speeds.



$\alpha_{12} = \alpha_{21} = 0$. In this case the diagram degenerates to the form shown in Fig. 5. When $I_D < I_P$, which is the case for a thin disc, the whirling speed is given by

$$\omega^2 = \frac{1}{\alpha_{11} M}$$

and is independent of γ. However when the rotor is a thick enough disc that $I_D > I_P$, then a second whirling speed occurs given by

$$\omega^2 = \frac{1}{\alpha_{22}(I_D - I_P)}$$

When $\gamma > 1$, $\alpha_{22}(I_D - I_P) > \alpha_{11} M$ and so then the second whirling speed is lower than the first. Therefore whirling can occur at a speed slower than

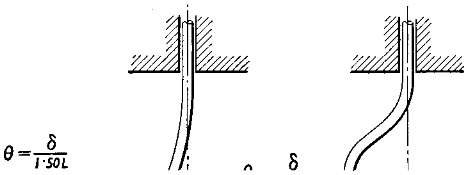
$$\omega^2 = \frac{1}{\alpha_{11} M}$$

on account of the gyroscopic moments provided by the spinning rotor. This is an important result which is not well known.

Example.—As a specific example of the calculation procedure, consider the system shown in Fig. 1. If the rotor is solid, and of mass M,

$$I_P = \frac{1}{8} M D^2$$

$$I_D = \frac{1}{16} M D^2 + \frac{1}{12} M l^2$$



Neglecting the usually small effect of gravity, the influence coefficients may easily be found from the theory of the bending of beams. α_{11} is the deflection of the centre of mass due to a horizontal force of unit magnitude acting on it. α_{22} is the angular deflection of the rotor (angle θ in Fig. 1) due to a unit moment on the rotor. $\alpha_{12} = \alpha_{21}$ is the angular deflection of the rotor due to the unit horizontal force, or the horizontal deflection of the mass centre due to the unit moment. In this case

$$\alpha_{11} = \frac{L^3}{EI}\left\{\frac{1}{3} + \frac{l}{2L} + \frac{l^2}{4L^2}\right\}$$

$$\alpha_{12} = \alpha_{21} = \frac{L^2}{EI}\left\{\frac{1}{2} + \frac{l}{2L}\right\}$$

$$\alpha_{22} = \frac{L}{EI}$$

where EI is the bending stiffness of the shaft. To keep the algebra simple, consider the case for which $l/2 = L = D$. Then

$$\gamma = \frac{(I_D - I_P)\alpha_{22}}{M\alpha_{11}} = \frac{13}{112}$$

$$\beta = \frac{\alpha_{11}\alpha_{22}}{\alpha_{11}\alpha_{22} - \alpha_{12}^2} = 28$$

Hence the frequency Equation (4) becomes

$$\frac{(13)}{(112)(28)}\Omega^4 - \frac{125}{112}\Omega^2 + 1 = 0$$

whose solutions are

$$\Omega^2 = 0{\cdot}895 \text{ or } 268$$

The two whirling speeds for this rotor are therefore given by

$$\omega = \frac{\Omega}{\sqrt{\alpha_{11} M}} = \frac{0{\cdot}945}{\sqrt{\alpha_{11} M}} \text{ or } \frac{16{\cdot}4}{\sqrt{\alpha_{11} M}}$$

which may be calculated numerically as soon as and M are specified. It is interesting

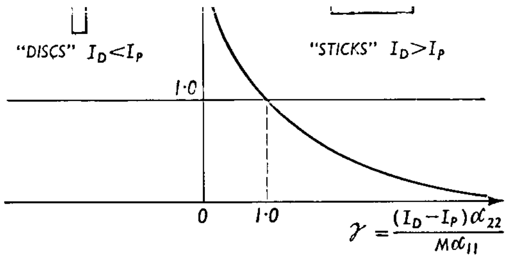


Fig. 5—Critical speeds for the special case of a symmetrically supported rotor ($\alpha_{12}=\alpha_{21}=0$)

A particularly interesting case occurs for a symmetrically supported rotor, in which

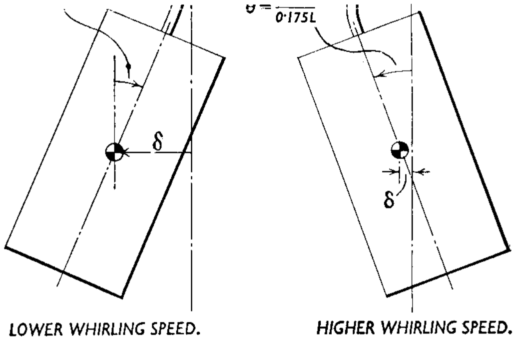


Fig. 6—Different mode shapes for the two whirling speeds

as α_{11} and M are specified. It is interesting to compare the two different mode shapes corresponding to these two whirling speeds. The ratio δ/θ may be obtained from either Equation (1) or Equation (2). The result of this calculation is that $\delta/\theta L=1{\cdot}50$ at the lower whirling speed and $\delta/\theta L=-0{\cdot}175$ at the higher speed. These two mode shapes are shown in Fig. 6.

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