Vibration of bell towers excited by bell ringing — a new approach to analysis

Richard Smith, Hugh Hunt
Cambridge University Engineering Department, Trumpington Street, Cambridge, CB2 1PZ.
e-mail: **hemh1@cam.ac.uk**

Abstract

Bell ringing is popular worldwide, especially in churches. In change ringing, especially popular in the UK, as many as twelve bells swing bodily through $360°$ and the mass of the heaviest bell is typically well in excess of 1000 kg. Such bells are generally located some way up a bell tower and the horizontal forces generated by their swinging can cause substantial—and often damaging—levels of vibration at frequencies typically around 2 Hz. Even small amplitudes of vibration, say 2 mm, are known to create difficulties because of the precise timing required to ring changes to a given “method”. In addition, a peal lasts roughly 3 hours 20 minutes and energy to vibrate the tower is energy extracted from the bell ringers’ muscular effort leading more rapidly to exhaustion.

This paper outlines a new method for analysing the motion of a tower during bell ringing. Traditionally it has been considered sufficient to keep odd harmonics of the fundamental period of the bell away from resonances of the tower and to avoid placing bells high up in towers with low-damped resonances in the vicinity of 1–2 Hz. The new method is much more precise than this and it examines the forces of all bells as they progress through a given method. Based on some simple data measured from the tower it is possible to compute the amplitude of tower motion with some considerable accuracy. It is found that even harmonics excite tower motion significantly and that this is related to the size of the handstroke gap. It is also found that total peal time has a big influence on tower motion. The paper presents a case study on the vibration of the bell tower in Great St Mary, Cambridge, which has been home to bell ringing since 1724 — the second oldest ringing society at any church in the world with a continuous history of its ringing.

1 Introduction

Many churches, both new and old, are fitted with rings of bells and Frost [1] gives an excellent account of the history of bell ringing in England. If rung in the “English Style” bells swing full-circle and the acoustical and dynamical characteristics of the bell tower are tested to their limits. In early times bells were swung gently, through a small arc, in a haphazard sequence. But by the 17th century in England it was common to find bells capable of swinging through $360°$ in a well-defined and musical sequence. Many bell towers had not been designed for the large horizontal forces that are generated and to this day bells in some towers cannot be rung for fear of serious damage or collapse. Frost gives a good account of the nature of the forces produced by swinging bells and of the resulting tower oscillation, but this account is insufficient for detailed design of bell towers and bell installations. A complete analysis of the forces generated is given by Heyman and Threlfall [2] but this treatment does not account for the motion of the tower. Bachmann et al. [3] give a mathematical account of the motion of bells and towers and they emphasise the importance of higher harmonics in the excitation of the tower. However their analysis concludes that the third harmonic is of greatest importance. This may be so for bell ringing in continental Europe, but for ringing in the English style it is the 7th, 9th, 11th and even the 13th harmonic of the fundamental swinging frequency that is driving the motion of the bell tower. Robinson and Windsor [4] give a description of the full analysis required to determine this motion and they recognize fully the importance of the higher harmonics. They also recognize the need to make good measurements of tower properties. Yet no simple procedure exists to compute the motion of a tower throughout the full three-hour-twenty-minute duration of a "peal". The procedures proposed until now have been complex, involving the time-domain solution of non-linear equations. This paper takes the Robinson–Windsor procedure to a logical conclusion by proposing a complete solution for bell-tower motion based on a fully-linear analysis where the tower is considered to be subject to a sequence of impulses. The computation time for a complete analysis is modest.

2 Equations of motion

A bell can be treated as a compound pendulum,[1] pivoting freely on gudgeon pins, which are positioned in bearings attached firmly to a frame. In this model, the frame is assumed not to move relative to the tower; however, the effect of tower sway is included by treating the tower as a damped oscillator moving horizontally.

The equations of motion of this system are readily found from the Lagrangian via the principle of least action. The tower's kinetic energy is $T_T = \frac{1}{2}M\dot{x}^2$, and its potential energy, $U_T = \frac{1}{2}\kappa x^2$, where x is the horizontal displacement of the tower, M the effective mass of the tower (including the mass of the bells), and κ its spring constant. Treating the tower as a closed system, the Lagrangian is

$$\mathcal{L}_T = T_T - U_T = \tfrac{1}{2}M\dot{x}^2 - \tfrac{1}{2}\kappa x^2. \tag{1}$$

An application of the Euler-Lagrange equation, gives $M\ddot{x} + \kappa x = 0$, the familiar equation for simple harmonic motion with frequency $\omega_T^2 = \kappa/M$. Clearly, a church tower is actually a damped system, and is expected to exhibit damped harmonic motion with damping constant λ,

$$\ddot{x} + 2\lambda\dot{x} + \omega_T^2 x = 0. \tag{2}$$

To handle damped motion in Lagrangian mechanics, the usual approach (see, for example, §25 of [5]) is to add a dissipative term to the Euler-Lagrange equation,

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} - \frac{\partial\mathcal{L}}{\partial x} = -\frac{\partial\mathcal{F}}{\partial\dot{x}}, \tag{3}$$

where $\mathcal{F}$ is the dissipative function. (Physically, $\mathcal{F}$ is half the rate of energy loss of the system.) By setting $\mathcal{F} = M\lambda\dot{x}^2$, the equations for damped harmonic motion can be derived.

With the tower at rest, the kinetic energy of the bell is straightforward, $T_B = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}mr_c^2\dot{\theta}^2$, where I is the moment of inertia of the bell about its centre of mass, m is its mass, r_c is the distance from the gudgeons (pivot) to the centre of mass, and θ the angular displacement of the bell from vertically down. It is convenient to rewrite inertial term in terms of the radius of gyration, r_g, via the definition $I = mr_g^2$. The potential energy is $U_B = -mgr_c\cos\theta$, where g is the acceleration due to gravity. This gives a Lagrangian for the bell as a closed system of

$$\mathcal{L}_B = T_B - U_B = \tfrac{1}{2}mr_g^2\dot{\theta}^2 + \tfrac{1}{2}mr_c^2\dot{\theta}^2 + mgr_c\cos\theta. \tag{4}$$

Applying the Euler-Lagrange equation (Eqn. 3) to this gives $m\left(r_g^2 + r_c^2\right)\ddot{\theta} + mgr_c\sin\theta = 0$, the familiar equation of motion of a pendulum with effective length,

$$l = \frac{r_g^2 + r_c^2}{r_c}. \tag{5}$$

[1]The term compound pendulum is sometimes misused to refer to a double or multiple pendulum. The term is used here in its more common meaning of a rigid body on a pivot, rather than a point mass on a massless string.

If the tower is not at rest, the kinetic energy of the bell gains a horizontal contribution from the tower motion, $\frac{1}{2}mr_c^2\dot{\theta}^2 \rightarrow \frac{1}{2}mr_c^2\dot{\theta}^2 + mr_c\dot{\theta}\dot{x}\cos\theta + \frac{1}{2}m\dot{x}^2$. Combining the two Lagrangians (Eqns. 1 and 4) with this coupling term gives the Lagrangian for the whole system,[2]

$$\mathcal{L} = \tfrac{1}{2}M\dot{x}^2 - \tfrac{1}{2}\kappa x^2 + \tfrac{1}{2}mlr_c\dot{\theta}^2 + mr_c\dot{\theta}\dot{x}\cos\theta + mgr_c\cos\theta. \tag{6}$$

As before, application of Euler-Lagrange (Eqn. 3) gives the equations of motion, of which this time there are two, one for x and one for θ.

$$\ddot{x} + 2\lambda\dot{x} + \omega_T^2 x = -\frac{mr_c}{M}\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right), \tag{7}$$

$$l\ddot{\theta} + g\sin\theta = -\ddot{x}\cos\theta. \tag{8}$$

These equations are governed by six[3] parameters: three describing the tower, M, λ and κ; and three describing the bell, l, r_c and m.

3 Measurement of parameters

The three parameters describing the dynamics of a bell are:

m	The mass of the bell
r_c	The distance from the bell's pivot to its centre of mass
l	The effective length of the bell when treated as a pendulum

The mass of each of the bells at Great St Mary's is recorded in [6]. The distance to the centre of mass of the bell can be determinted from the the wheel radius, measured out to the centre of the rope, r_w, together with the angular deflection, α, of the bell when a known weight, W, is on the rope. Resolving the torque on the wheel gives,

$$mgr_c\sin\alpha = W\,r_w. \tag{9}$$

The final parameter, l, can be determined from the bell's small swing period, $\tau_0 = 2\pi/\omega_0 = 2\pi\sqrt{l/g}$.

Bell	m (kg)	r_c (mm)	τ_0 (s)	ω_0 (s^{-1})	l (mm)	r_g (mm)	I ($\mathrm{kg\,m^2}$)
1	264	348	1.56	4.03	605	299	23.57
2	315	336	1.57	3.99	615	306	29.54
3	278	407	1.59	3.95	628	300	24.95
4	303	405	1.64	3.84	665	325	32.01
5	345	417	1.64	3.84	665	322	35.73
6	369	429	1.69	3.71	712	349	44.87
7	411	417	1.74	3.61	752	374	57.43
8	518	428	1.78	3.52	790	394	80.27
9	659	403	1.86	3.38	856	427	120.31
10	731	438	1.89	3.32	887	444	143.91
11	1030	451	1.99	3.16	984	490	247.49
12	1376	559	2.03	3.09	1027	511	360.04

Table 1: The parameters of the bells at Great St Mary's

The three parameters describing the dynamics of a tower are:

[2] A similar example is given in §5 of [5].

[3] Seven if the acceleration due to gravity, g, is included; but unless ringing on the Moon is of interest, g will not vary significantly.

λ	The damping constant of the tower
κ	The spring constant of the tower
M	The effective mass of the tower

Measuring the parameters of the tower requires setting up accelerometers in the tower, level of the bells, and taking measurements while the bells are rung. This has been done on two occasions, first by Windsor, and then independently by the authors. Both sets of measurements used similar methodologies: steadily ringing a bell up or down and recording the tower deflection in order to locate the resonant frequency of the tower, ω_R; and recording the speed at which the tower oscillations die away to determine λ.

When the tower is resonating, it is necessary to identify which resonance is occuring. Assuming several resonances identified, which resonances were found rapidly becomes apparent from the ratio of the speeds at which be the bell is being rung; for example, if the bell's time periods, τ, are in the ratio 5 : 7 : 9, then the 5th, 7th and 9th harmonics were found. The tower's resonant frequency f_R is then n/τ.

In exponential decay, the time taken to decay to e^{-1} of the original value is λ^{-1}. This can be measured straightforwardly from the acceleratormeter data. Because the tower is buttressed by the nave to the east, it is stronger in that direction. Consequently, each parameter potentially has different values for E–W movement and N–S movement. The values taken in the two sets of tower measurements are shown in Table 2.

	Source	f_R (Hz)	ω_R (s^{-1})	λ^{-1} (s)	Q
N–S	W	1.71	10.7	6.5	35
	A	1.68	10.6	7	37
E–W	W	1.95	12.3	3.6	22
	A	1.91	12.0	4	24

Table 2: The frequency and Q-factor of the tower as measured by Windsor ('W') and by the authors ('A').

4 Motion of the bell

The term on the right of the equation of motion for θ (Eqn. 8) is the force that causes a bell to drop or fly unexpectedly over the balance. However, despite being of considerable relevance, it will be ignored for the moment. This can be physically justified by asserting that a sufficiently competent ringer will apply an equal and opposite force to the bell so that the bell will sound at the intended moment, despite tower motion. As such, whilst removing the term is an approximation, it is likely a better approximation to leaving it in. This leaves an homogeneous second order differential equation,

$$\frac{d^2\theta}{dt^2} + \omega_0^2 \sin\theta = 0, \tag{10}$$

where $\omega_0 = \sqrt{g/l}$, the small amplitude frequency. The solution to this equation is well known (see e.g. §2.571 of [7]),

$$\omega_0 t = \int_{\pi/2}^{\phi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}} = F(\phi, k) - K(k) \tag{11}$$

where $F(\phi, k)$ is the incomplete elliptic integral of the first kind,[4] $K(k)$ is the corresponding complete elliptic integral, and k and ϕ are the result of substitutions, $\sin\theta/2 = k \sin\phi$ and $k = \sin\theta_0/2$. The lower

[4]There are many of alternative definitions and notions for elliptic integrals in common usage. (See §17.2 of [8].) The form used here is Legendre form,

$$F(\phi, k) = \int_0^{\phi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}}.$$

limit of the integral corresponds to the boundary condition of the bell being at an angle θ_0 at $t = 0$ and gives rise to $K(k)$ term.

The integral equation (Eqn. 11), $u = F(\phi, k)$ where $u = \omega_0 t + K(k)$, can be inverted to express ϕ in terms of u by using the fact that elliptic functions are inverses of elliptic integrals,[5]

$$\sin\phi = \mathrm{sn}(u, k). \tag{12}$$

The time period of $\mathrm{sn}\, u$ is $\tau = 4K/\omega_0$, or equivalently, the angular frequency is $\omega_B = \pi\omega_0/2K$.

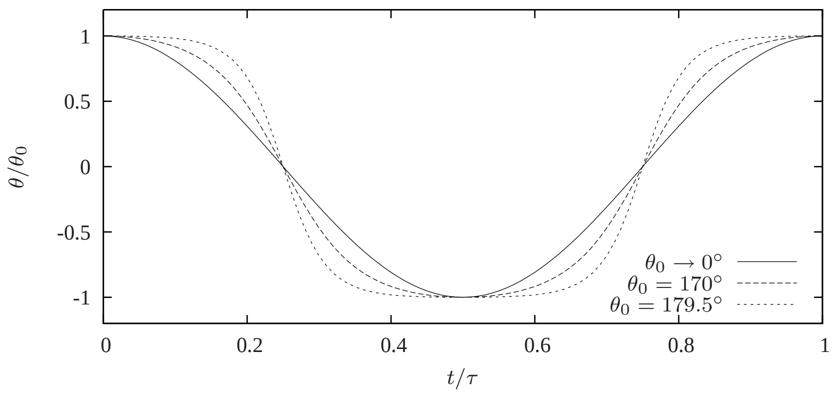

Figure 1: Angle of bell throughout a whole cycle for various values of θ_0.

In the limiting case of very small swings, $\theta_0 \to 0$ and therefore $k \to 0$. Applying a small angle approximation, $\phi = \mathrm{sn}(u, k) \approx \sin u$, restores the familiar small angle formula, $\theta = \theta_0 \cos\omega_0 t$. As the bell is rung right up to the balance, $\theta_0 \to \pi$, $k \to 1$ and $K(k) \to \infty$. This means that by ringing the bell closer to the balance, it can be made to ring arbitrarily slowly, as is indeed the case. This can been in the flattening of the graphs for larger θ_0 in Figure 1.

5 Motion of the tower

With no force from a bell, the tower acts as a damped harmonic oscillator (Eqn. 7, with the right hand side removed),

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega_T^2 x = 0. \tag{13}$$

This has solution

$$x = Ae^{-\lambda t}\sin(\omega_T' t + \delta) \tag{14}$$

where $\omega_T'^2 = \omega_T^2 - \lambda^2$. (In practice, $\lambda \ll \omega_T$ so $\omega_T' \approx \omega_T$.) A, the amplitude, and δ, the phase shift, are constants of integration. With a forcing function included, the equation of motion becomes the inhomogeneous equation derived earlier (Eqn. 7) which can be expressed,

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega_T^2 x = \frac{f(t)}{M}, \tag{15}$$

where $f(t)$ is the force applied to the tower by the bell's motion,

$$f(t) = 2kf_0\left(1 + 4k^2 - 6k^2\,\mathrm{sn}^2\,u\right)\mathrm{sn}\,u\ \mathrm{dn}\,u. \tag{16}$$

and where $f_0 = mr_c\omega_0^2$.

[5]It is customary to omit the second argument, k, to elliptic functions when it is clear from context—that is $\mathrm{sn}(u, k)$ is often just written $\mathrm{sn}\,u$. Similarly $K(k)$ is often abbreviated to K.

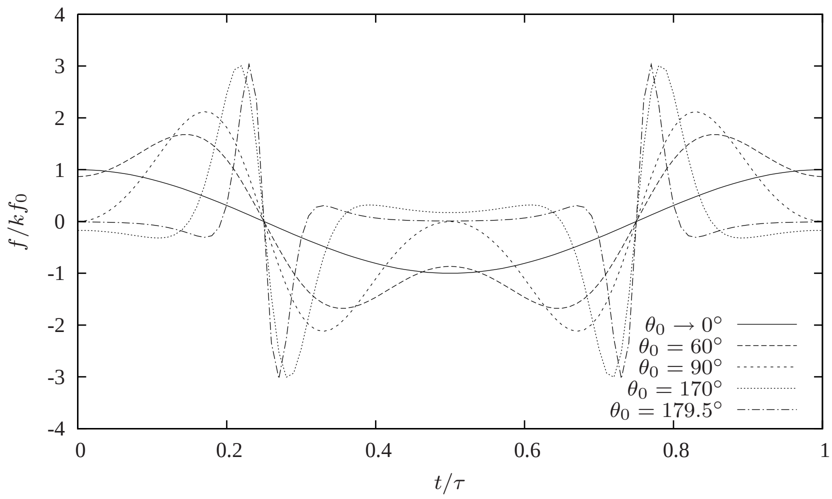

Figure 2: Driving force as a function of time for various values of θ_0.

For small swings, $\operatorname{sn} u \approx \sin u$ and $\operatorname{dn} u \approx 1$ making the force cosinusoidal: $f = f_0\theta_0 \cos \omega_0 t = f_0\theta$. At $t = 0$, the bell is at maximum angle, and the force is at a maximum too. From Figure 2 it can be seen that when the bell swings through larger angles the initial force is small (compared to later in the cycle)—intuitively, when the bell is slowly moving near the balance, it will produce very little force on the tower.

Further analysis (such as in §4 of [4]) allows f to writen as a complex Fourier series

$$f(t) = f_0 \sum_{n=-\infty}^{\infty} c_n e^{in\omega_B t}, \tag{17}$$

with coefficients,

$$c_n = (1-(-1)^n)\frac{\pi i^n}{K}\left(\frac{in\pi}{2K}\right)^3 \frac{q^{n/2}}{1+q^n}, \tag{18}$$

and where $q = \exp(-\pi K'/K)$. Clearly this vanishes for even n, and for odd n the coefficient is real valued. Also, as $c_n = c_{-n}$, the complex exponential Fourier series can be converted into a cosine Fourier series,

$$f(t) = f_0 \sum_{n=0}^{\infty} 2c_{2n+1} \cos(2n+1)\omega_B t, \tag{19}$$

Numeric evaluation of c_n requires care as $k \to 0°$ and as $k \to 180°$, as K' and K diverge as those limits are approached. Figure 3 shows that as $\theta_0 \to 180°$, more terms in the Fourier series become relevant. For example, at $\theta = 120°$ the 3rd harmonic dominates, with significant contributions from the 1st and 5th.

6 Tower–bell resonance

With the forcing function expressed as a Fourier series, the equation of motion (Eqn. 15) can now be solved in terms of the Fourier components.

$$x = \sum_{n=0}^{\infty} \frac{2f_0\, c_{2n+1}}{\omega_T^2 M}\, \Omega_{2n+1}\, S_{2n+1} \tag{20}$$

where

$$S_n = \sin(n\omega_B t + \delta_{Bn}) - \frac{\omega_T}{\omega_T'} e^{-\lambda t} \sin(\omega_T' t + \delta_{Tn}), \tag{21}$$

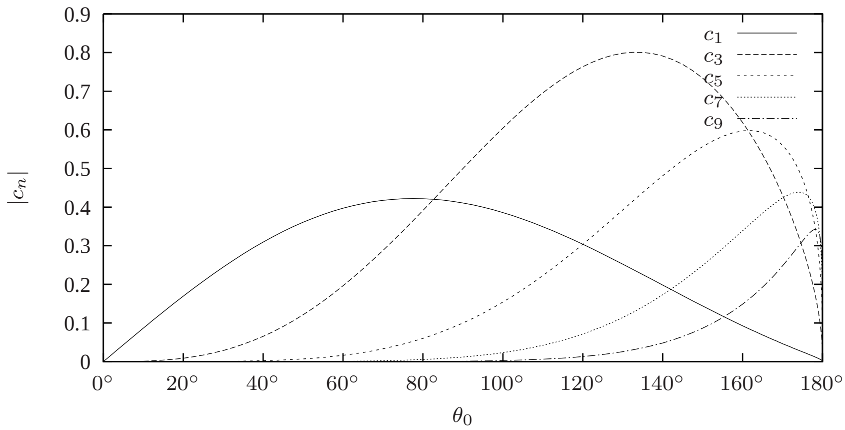

Figure 3: The relative magnitudes of first five Fourier coefficients.

and

$$\Omega_n = \frac{\omega_T^2}{\sqrt{(\omega_T^2 - n^2\omega_B^2)^2 + 4\lambda^2 n^2 \omega_B^2}}. \tag{22}$$

The exact values of the phase shifts, δ_{Bn} and δ_{Tn} are not relevant to this analysis.

The second term in Eqn. 21 decays exponentially over time (due to the $e^{-\lambda t}$ term), and therefore reflects some transient resulting from the initial boundary condition, $\dot{x} = x = 0$ at $t = 0$. When considering the steady state effect of ringing a bell continuously, this term vanishes. After the transients have died away, the frequency-dependence of the amplitude for a given harmonic is given almost entirely[6] by Ω_n.

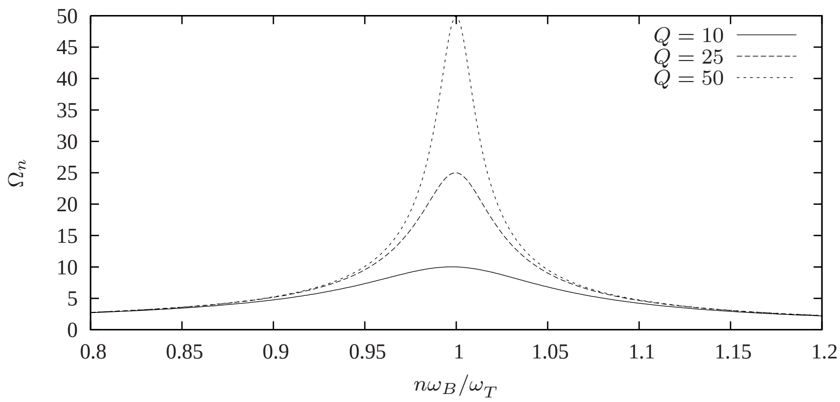

Figure 4: Amplitude at various frequencies around resonance.

The amplitude, Ω_n, reaches a maximum when $n\omega_B = \omega_R$, where ω_R is the resonant frequency of the tower, $\omega_R^2 = \omega_T^2 - 2\lambda^2$. This is shown in Figure 4 for several values of λ, where λ is expressed in terms of the dimensionless Q-factor, $Q = \omega_T/2\lambda$. From this graph it is apparent that at resonance, $\Omega_n \approx Q$. It can be verified that when $n\omega_B = \omega_T$ (i.e. very near to, but not at, the resonance peak) this approximation is exact.

Away from resonance, as $\omega_B \to \infty$ (i.e. for much higher harmonics), $\Omega_n \to 0$; and as $\omega_B \to 0$ (i.e. for lower harmonics), $\Omega_n \to 1$. The amplitude of a given harmonic is given by

$$A_n = \frac{2f_0\ |c_n|\ \Omega_n}{M\omega_T^2}, \tag{23}$$

[6] There is also frequency dependency in c_n as it depends on k and hence indirectly on ω_B; this introduces a very small correction to the resonant frequency. The correction is small because c_n is a much flatter function of ω_B than Ω_n is.

and the overall amplitude is $A = \sum_n A_{2n+1}$. For resonance to be a significant physical effect, one one particular A_n (the resonant amplitude) must dominate this sum. This requires $Q\, c_{2n+1}$ to be significantly larger than the other coefficients.

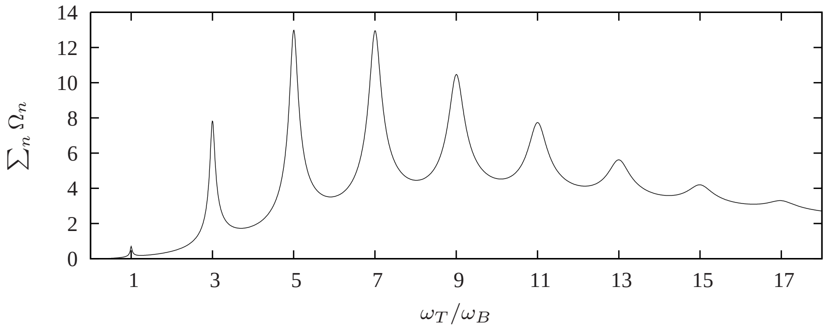

Figure 5: The first nine resonances for a bell swinging at $\theta_0 = 175°$ in a tower with $Q = 25$.

The only time-dependence in x comes from S_n,

$$S_{2n+1} = \sin(\omega_{Bn} t + \delta_{Bn}) - \frac{\omega_T}{\omega'_T} e^{-\lambda t} \sin(\omega'_T t + \delta_{Tn}). \tag{24}$$

At resonance, with $Q \gg 1$, both phase shifts (δ_{Tn} and δ_{Bn}) are approximately the same; similarly, for physically realistic systems, ω_T, ω'_T and ω_R are all approximately the same. This means that at resonance, when $n\omega_B = \omega_R$, S_n can be simplified to

$$S_n = (1 - e^{-\lambda t}) \sin(\omega_T t + \delta) + \mathcal{O}(Q^{-2}). \tag{25}$$

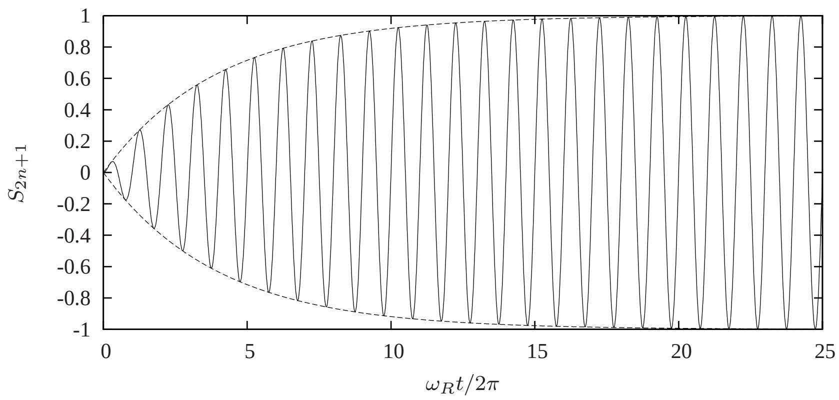

Figure 6: Initial transients at resonance with $Q = 25$. The dotted line is the envelope $\pm(1 - e^{-\lambda t})$.

An alternative way of studying the tower transients, at least at the resonance harmonic, is to look at the impulse response from one half cycle of the bell,

$$R_n e^{-\lambda t} \sin(\omega_T t + \delta). \tag{26}$$

The transients while resonance is building up is then the series,

$$S_n = \sum_{r=1}^{\lfloor 2t/\tau \rfloor} (-1)^r R_n e^{-\lambda(t - r\tau/2)} \sin\big(\omega_T (t - r\tau/2) + \delta\big). \tag{27}$$

The physical interpretation of this is as a superposition of impulse responses at intervals of $\tau/2$ at the end of every half cycle of the bell (corresponding to every time the bell strikes).[7] The impulse in one half of the cycle is equal and opposite to the impulse in the other half, hence the $(-1)^r$.

It was established in §5 that resonance only occurs at the odd harmonics. This can also be seen from Eqn. 27 by noting that $r\tau\omega_T/2 = rn\pi$. This means that for odd n, consecutive impulses superpose constructively. Summing the series then gives,[8]

$$S_n = R_n \frac{1 - e^{-\lambda t}}{1 - e^{-\lambda\tau/2}} \sin(\omega_T t + \delta) \tag{28}$$

and by comparison with Eqn. 25,

$$R_n = 1 - e^{-\lambda\tau/2} = 1 - e^{-n\pi/2Q}. \tag{29}$$

n	1	3	5	7	9	11
N–S	24.3	8.44	5.28	3.92	3.18	2.70
E–W	15.7	5.58	3.57	2.71	2.24	1.94

Table 3: The first few values of R_n^{-1}

7 Impulse responses

The impulse from a single bell struck at time $t = 0$ will cause the tower to oscillate in damped simple harmonic motion,

$$x = \xi \, \mathrm{e}^{-\lambda t} \sin \omega_T t \tag{30}$$

where ξ is the initial amplitude. The cumulative effect of a single bell ringing steady blows with period $\frac{1}{2}\tau$ (i.e. whole pulls with period τ) has the amplitude $R_n^{-1}\xi$ To analyse the effect of a single bell, the authors measured the tower displacement, x, while a bell was gradually rung faster and faster (i.e. it was slowly rung down). At various points, the frequency at which the bell was being rung would reach a resonance with the tower motion and significantly larger oscillations were detected. From this, ξ can be calculated.

Bell	x_7 (mm)	ξ (mm)	x_5 (mm)	ξ (mm)
11	0.62	0.16	0.76	0.14
10	0.44	0.11	0.57	0.11
7	0.27	0.07	0.30	0.06
3	0.25	0.06	0.31	0.06
2	0.19	0.05	0.26	0.05
1	0.18	0.05	——	

Table 4: North–south oscillation amplitudes

For north–south oscillations, the two values of ξ calculated per bell are in moderately good agreement; this is slightly less true for east–west oscillations, especially for the 5th harmonic. In both cases, the lower

[7]The reason for putting the impulse at the end of the bell's half cycle is so that the tower motion is alway underestimated rather than overestimated. Whether this is the correct decision will depend on its intended use.

[8]Eqn. 27 is only exactly equal to Eqn. 28 when $\lambda\tau/2 \cdot \lfloor 2t/\tau \rfloor = \lambda t$ which is true when $t = r\tau/2$ for integer r.

Bell	x_9 (mm)	ξ (mm)	x_7 (mm)	ξ (mm)	x_5 (mm)	ξ (mm)
12	0.37	0.17	0.45	0.17	———	
9	0.22	0.10	0.21	0.08	0.14	0.04
8	0.17	0.08	0.22	0.08	———	
6	0.16	0.07	———		0.10	0.03
5	———		0.13	0.05	———	
4	———		0.16	0.06	———	

Table 5: East–west oscillation amplitudes

harmonics result in smaller values of ξ. This is to be expected as the bell is lower down for the lower harmonics, and thus has less energy available to drive oscillations in the tower.

For the remainder of this analysis, the values of ξ derived from the highest harmonics given in Tables 4 and 5 are used. These are summarised in Table 6. The sign of ξ in this table indicates the relative directions in the bells swing. For example, the 10th and 11th swing in opposite directions. The actual choice of sign is arbitrary.

Bell	1	2	3	7	10	11
ξ	$+0.05$	$+0.05$	$+0.06$	-0.07	-0.11	$+0.16$
Bell	4	5	6	8	9	12
ξ	$+0.06$	$+0.05$	$+0.07$	-0.08	$+0.10$	$+0.17$

Table 6: Values of ξ used for the following analysis

8 Modelling ringing rounds

The effect of change ringing can be simulated by superposing damped sine waves (Equation 30) for each blow made by a bell, with a factor of ± 1 depending on whether it is at handstroke or backstoke. When considering ringing rounds on N bells, the bells are assumed to sound perfectly rhythmically in the ascending order of size (from 1 to 12) with a spacing of $\tau/(2N+\delta)$ between bells where δ is the size of the handstroke gap.[9] In ordinary ringing, δ is usually reckoned on being 1, and when 'cartwheeling', $\delta = 0$; unless otherwise stated, $\delta = 1$. The north–south and east–west oscillations are summed independently, and are then combined to get the magnitude,

$$|\mathbf{x}| = \sqrt{\left(\sum x_{\text{NS}}\right)^2 + \left(\sum x_{\text{EW}}\right)^2}$$

Figure 7 show the results of numerical calculatio for eight whole pulls of rounds at peal speeds of 3h20. The peal speed is the length of time taken to ring 5000 "changes"—i.e. to ring each bell 5000 times. Interestingly, the tower sway is not the same at handstroke and at backstroke—once the oscillations have settled into a pattern, the graph indicates magnitudes of around 0.6–0.7 mm for the first half of each whole turn (handstroke), but only of around 0.3–0.4 mm in the second half (backstroke). This asymmetry can only be explained by the presence of the open handstroke lead, and a simulation with closed handstroke leads (Figure 8) no longer exhibits this asymmetry.

[9]The handstroke gap is a small pause left after each bell has rung twice; it acts as a form of musical phrasing.

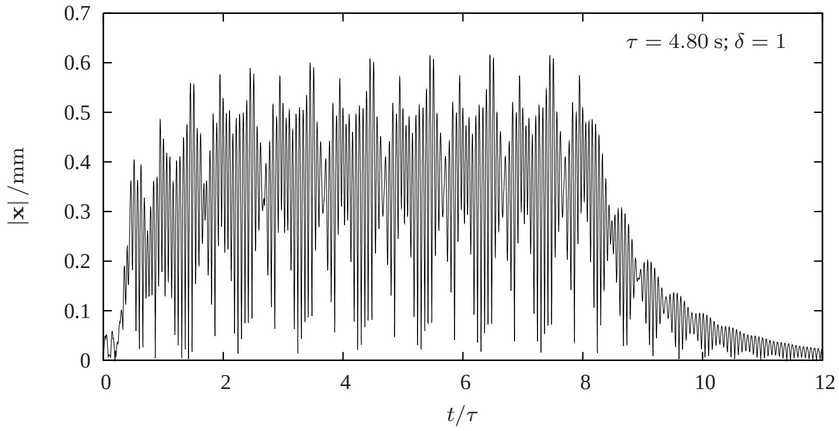

Figure 7: Magnitude of tower sway for eight whole pulls of rounds with open handstroke leads at a peal speed of 3h20

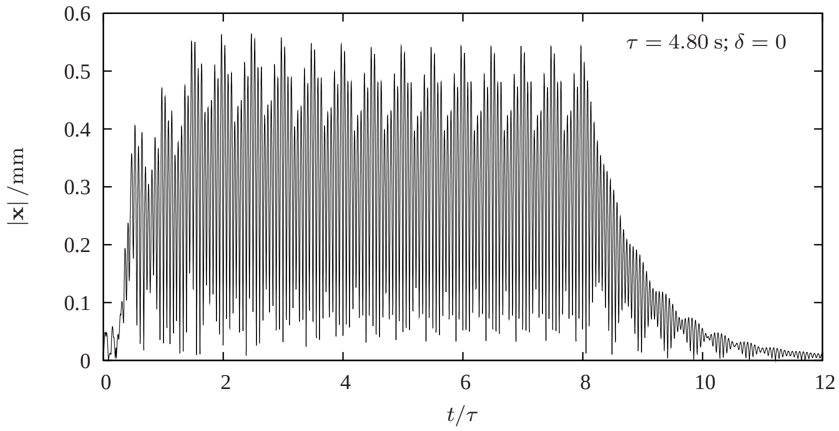

Figure 8: Magnitude of tower sway for eight whole pulls of rounds with closed handstroke leads at a peal speed of 3h20

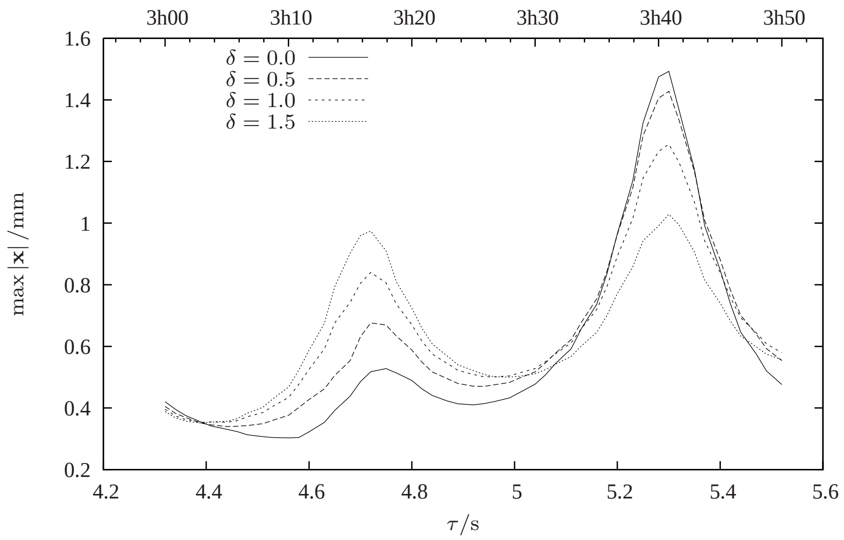

Figure 9: Maximum tower sway during rounds at different speeds

Whilst the variation in tower sway during a whole pull is of interest, the most important detail is the maximum sway. This is plotted in Figure 9.

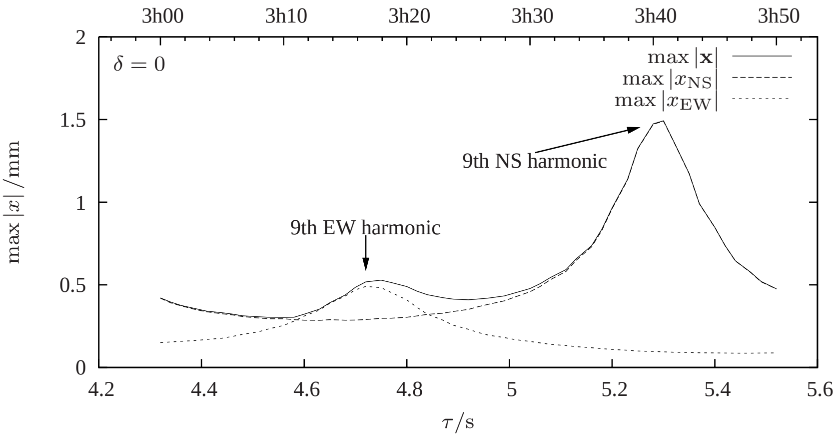

Figure 10: North–south and east–west components of tower sway at different speeds with closed handstroke lead

The peak at 3h41 ($\tau = 5.30$ s) can be explained as excitation of the 9th north–south harmonic at $\tau = 9/(1.7\ \mathrm{Hz}) = 5.29$ s; and Figure 10 confirms that this peak is primarily due to north–south motion. The fact that the peak is more pronounced for smaller δ bears this out—odd harmonic resonance relies on the handstroke and backstroke being equally spaced. The amplitude at the peak, 1.49 mm is also consistent with this this. Table 3 gives the value of $R_9^{-1} = 3.18$ and from Table 6, $\sum_{\mathrm{NS}} \xi = 0.50$ mm. If the bells were ringing at once, the amplitude of this peak would be expected to be $R_9^{-1} \cdot \sum \xi = 1.59$ mm, plus a small contribution from the east–west motion; in rounds, not all of the bells will be in phase with the tower oscillations, and the peak should be slightly smaller because of this.

The other peak, at about 3h17 ($\tau = 4.72$ s) is in the right position to be explained by the 9th east–west harmonic at $\tau = 9/(1.9\ \mathrm{Hz}) = 4.74$ s, but it is curious as it is stronger for *larger* δ. With closed handstroke leads (i.e. $\delta = 0$), the amplitude of this peak is 0.53 mm, and a quick estimate from $R_9^{-1} \cdot \sum \xi$ gives a prediction of a bit less than $2.24 \times 0.53\ \mathrm{mm} = 1.18\ \mathrm{mm}$. This is not in particularly close agreement, but it could be explained by the bells being further out of phase with the tower oscillation than for the 3h41 peak.

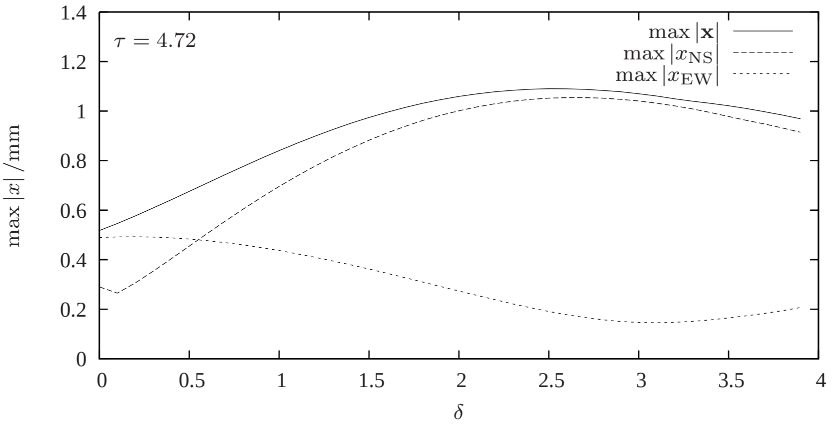

Figure 11: Tower sway during rounds with different sized handstroke leads at 3h17 peal speed

The fact that the 3h17 peak grows in amplitude with δ is of particular interest. From Figure 11 it can be seen that, whilst the peak at $\delta = 0$ is primarily due to east–west motion (the 9th harmonic), the additional contribution when $\delta > 0$ is from north–south motion. Curiously, 3h17 ($\tau = 4.72$ s) is also very close to the 8th north–south harmonic at $\tau = 8/(1.7 \text{ Hz}) = 4.71$ s. Only odd harmonics were expected to occur; however, this assumes entirely regular ringing—i.e. ringing with close handstrokes. Attributing the 3h17 peak to a combination of the 9th east–west harmonic and the 8th north–south harmonic would explain why the 3h17 peak is present at $\delta = 0$ and why it grows with δ.

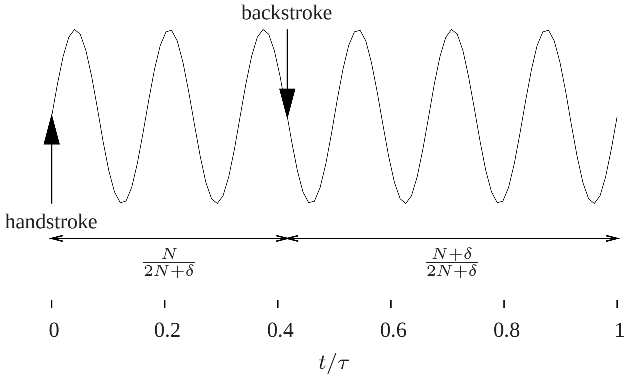

Figure 12: An even harmonic exciting by open handstroke leads

Figure 12 shows the way in which an even harmonic can be excited. From this it is apparent that the nth harmonic resonance (n even) occurs when the duration of the backstroke is one oscillation longer than the handstroke.

$$\frac{\delta\tau}{2N+\delta} = \frac{1}{f} \qquad \delta = \frac{2N}{f\left(\tau - \frac{1}{f}\right)} = \frac{2N}{n-1} \tag{31}$$

9 Modelling plain hunt on twelve

The logical next step is to look at a simple method, such as plain hunt on twelve bells (Figure 13; E, T represent bells 11 and 12). In this, bells spend most of their time "hunting up" (ringing every $N+1$ bells) or "hunting down" (ringing every $N-1$ bells). Effectively plain hunt is the superposition of two separate speeds of ringing. This results in two separate opportunities for exciting each tower harmonic, and instead of occuring when $\tau = n/f$, they occur at

$$\tau = \frac{2N+\delta}{2N+\delta \pm 2}\,\frac{n}{f} \tag{32}$$

Table 7 gives the values of τ for north–south resonances, with those corresponding to peal speeds of between 3h00 and 3h50 shown in bold.

Figures 14 and 15 show the modelled tower sway during plain hunt at 3h15 and 3h24 speed respectively. At 3h15, although the rounds cause significant sway, this quickly dies away during the plain hunt. At the slightly slower speed, this is no longer true, and during the second half of the course, there is significant tower sway. The second half of the course is when the 10th and 11th are hunting up and thus driving the 9th north–south harmonic resonance.

```
1234567890ET
2143658709TE
241638507T9E
42618305T7E9
4628103T5E79
648201T3E597
68402T1E3957
8604T2E19375
806T4E291735
08T6E4927153
0T8E69472513
T0E896745231
TE0987654321
ET9078563412
E9T705836142
9E7T50381624
97E5T3018264
795E3T102846
7593E1T20486
57391E2T4068
537192E4T608
3517294E6T80
31527496E8T0
132547698E0T
------------
1234567890ET
```

Figure 13: Sequence of bells in plain hunt

The maximum sway at various different peal speeds and sizes of handstroke lead is plotted in Figure 16. The four peaks coincide to four of the five bold figures in Table 7. (The missing one is the $\tau = 4.36$ 8th harmonic which is right on the edge of the region of study.) Interestingly, there are no obvious east–west harmonics being excited.

n	τ_{up}	τ_{down}
6	3.27	3.84
7	3.81	**4.48**
8	**4.36**	**5.12**
9	**4.90**	5.75
10	**5.45**	6.39
11	5.99	7.03

Table 7: Expected north–south resonances during plain hunt with $\delta = 1$

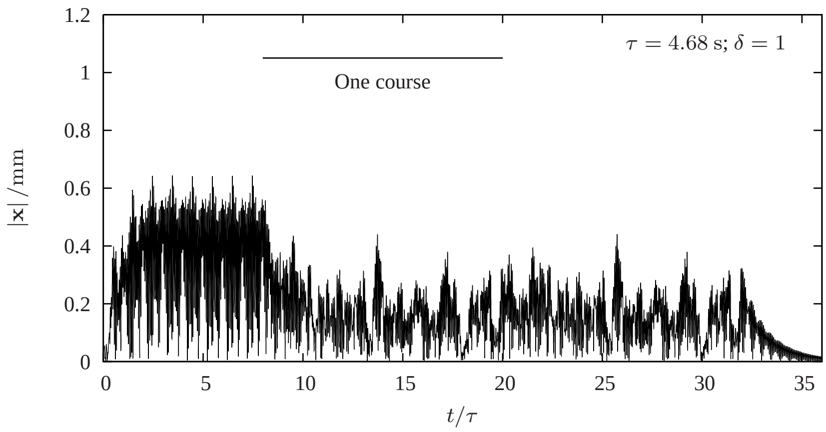

Figure 14: Eight whole pulls of rounds, followed by 48 changes of plain hunt on twelve, rung at 3h15 speed

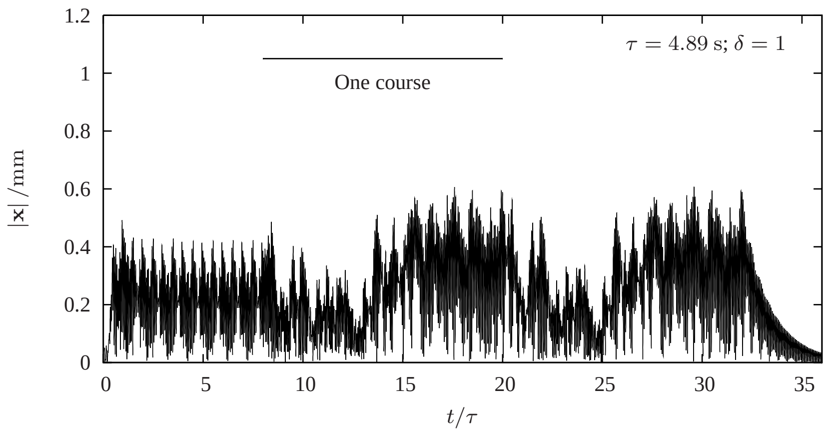

Figure 15: Eight whole pulls of rounds, followed by 48 changes of plain hunt on twelve, rung at 3h24 speed

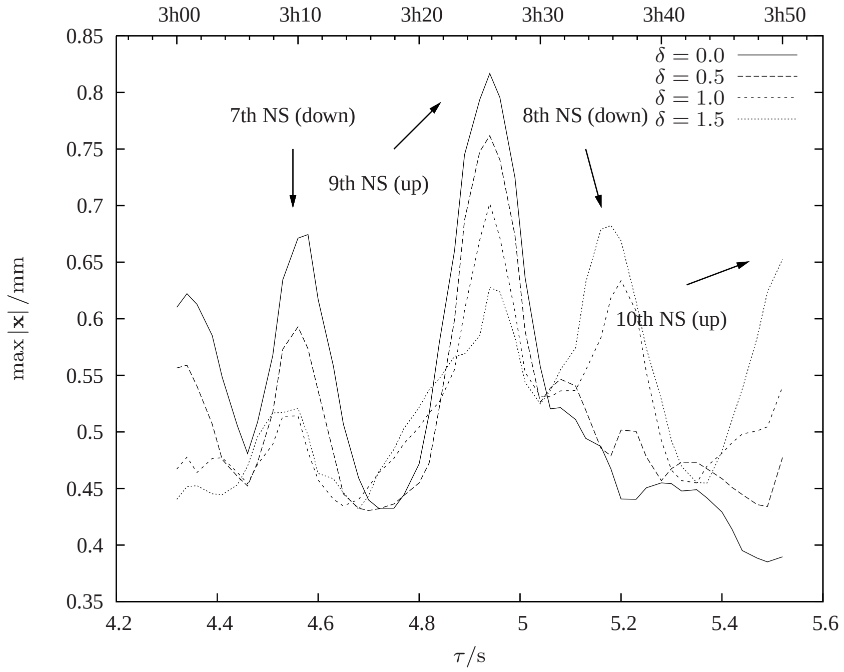

Figure 16: Maximum tower sway during plain hunt on twelve at different speeds

10 Conclusions

The procedure described above can be used to predict accurately the motion of bell towers given the characteristics of the bells, their alignment and the dynamic characteristics of the tower. All of these properties are easy to measure or to calculate. Tower motion is found to be dependent on the method and the amplitude varies significantly during the peal. Also of great importance is the peal time. A small change in peal time can lead to significant changes in tower amplitude. Other factors found to be significant are the handstroke gap. All of these can be analysed simply and accurately by assessing the response of the bell tower to a series of carefully-timed impulses.

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