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Torsional Behaviour Of Cello Strings

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Summary
The behaviour of a bowed string depends, among other things, on the frequency, impedance and internal damping of torsional waves on the string. Very little published information is available about these quantities, especially the torsional damping. Measurements of all relevant torsional properties have been made on cello strings of three different constructions. These show that the torsional modes are harmonically spaced to reasonable accuracy, and that the Q factors are approximately equal for all modes of a given string. These torsional Q factors are roughly an order of magnitude smaller than those of the transverse modes of the same string. The torsional wave speed varies somewhat with the tension in the string, decreasing with higher tension. The damping factors are not significantly influenced by tension. These results have been expressed in terms of a novel "reflection function" [1] suitable for direct incorporation into simulations of the bowing process.

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1. Introduction

When a string on a violin or cello is bowed, the behaviour is predominantly governed by transverse vibration of the string. The played frequency is ordinarily close to the fundamental transverse resonant frequency, and the radiated sound from the instrument body is driven mainly by the oscillating force at the bridge resulting from transverse string motion. However, the other types of travelling wave on the string cannot be ignored. The possible role of longitudinal waves is still somewhat controversial, but, surprisingly, torsional waves have been shown to have a significant influence. A bowed string is driven by frictional force from the bow hairs, and this force is applied tangentially to the surface of the string. Thus it excites both transverse and torsional waves, in a ratio of amplitudes governed by the ratio of transverse to torsional wave impedances [2].

The torsional waves are not in themselves responsible for much sound radiation by the instrument body, since they can only apply a rather small moment to the bridge, determined by the diameter of the string. However, they have a strong influence on the stability of periodic bowed-string motion, and on the length and pattern of transients. The reason is that torsional waves have very much higher damping than transverse waves (as will be shown shortly). This means that energy put into torsional waves at the bow is dissipated relatively quickly. The importance of this only emerged when the first attempts were made to simulate bowed-string motion from theoretical models [3]. If torsional motion was not included, and realistic parameter values were used to characterise the damping of transverse string motion, then it was found that periodic regimes of string vibration were almost invariably unstable. Including torsional motion with realistic damping levels was found to produce stability. This stability behaviour has subsequently been analysed in detail [4], and the effect confirmed. The conclusion is that it would probably not be possible to play steady, musical notes on a violin without the stabilising influence of torsional string motion!

In recent years the sophistication of simulation models has increased, and a goal for the next few years is to make detailed comparisons of simulated and measured bowing transients. If convincing agreement can be achieved, then the way will be open to investigate through simulation a number of important practical issues relating to string design and to differences in "playability" between instruments [1, 5]. An obvious prerequisite for such realistic simulations is to have reliable quantitative information about the frequency and damping of torsional waves on strings. Unfortunately, such information is not easy to obtain and very little is as yet available in print [2, 6, 7]. This note presents a modest addition to this meagre literature.

The reason torsional data is elusive is that it is far from easy to drive and observe torsional motion of a string without disturbing the behaviour significantly by the effects of attached sensors. The relatively high damping of torsional motion is also part of the problem. In a spectrum containing both transverse and torsional response, torsional peaks tend to be concealed by the much stronger, narrower and closer-spaced transverse peaks. The only method for driving and observing torsional motion which has been successful in the past has involved gluing a loop of fine wire to a few centimetres of the string, up one side and back down the other. When this loop is placed in a uniform magnetic field, oriented in the plane of the loop, a voltage is generated across the wire proportional to differential velocity on the two sides of the string, in other words to torsional angular velocity (for small amplitudes of motion). This device can be used as a sensor and/or a driver. Both Cremer [6] and Gillan and Elliott [7] used this device for their measurements. As they were aware, there is a danger of modifying the string behaviour by the attached mass of the wire loop, and also by the glue used to attach it. Nevertheless, for a reasonably thick string the method gives credible results.

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The present investigation was initially prompted by the availability of a new optical sensor, developed for use in the Atomic Force Microscope [8]. This works on the familiar principle of the "optical lever". A small mirror, which can be a 1 mm square of cleaved mica with a gold-coated surface, is fixed to the object to be monitored, in our case the surface of the string. A laser beam is shone on this mirror at an oblique angle. The reflected beam, whose precise position will vary if the mirror rotates, falls on a sensor consisting of four photosensitive patches arranged as quadrants of a circle. These sensors can be wired up in various differential configurations, to respond with varying degrees of sensitivity to movements of the reflected beam.

Since the small mirror is less intrusive than the wire loop sensor, it was originally hoped to make measurements using transient excitation of the string which would allow torsional motion to be observed with minimal disturbance to the string's mechanical properties. Unfortunately, we were not able to devise a method of excitation which did not suffer from the problem of "swamping" of the torsional signal by effects of transverse string motion. To obtain reasonable results we were forced to compromise, and use a wire-loop exciter and the optical sensor.

2. Measurement of torsional modes

Measurements were made on three cello strings of different construction (described later). These were thick enough that one would not expect significant disturbance to be caused by the mirror or the wire loop. Each string was mounted between rather rigid "bridges" fixed to a heavy optical table. The bridges were placed 700 mm apart, close to the string length of a cello (typically about 690 mm). The string was tensioned so that the fundamental transverse resonance reached its nominal pitch. The wire loop was glued to the string with a thin layer of cyanoacrylate adhesive, very close to one end of the string so that the wires could be carried over the bridge before making arrangements for electrical connections. Permanent magnets were placed on either side of the loop, and the loop was driven with an input signal of the required waveform, via a current amplifier so that the applied torque follows the input signal. Near the other end of the string the mica mirror was attached, and the laser and sensor were adjusted to respond to torsional displacement. The total added mass of wire loop, mirror and glue was only about 2 mg, small enough to be negligible. Of slightly more concern, there is a possibility that the glue, by getting between the winding of the string, will have locally modified the torsional stiffness somewhat. However, the amount of glue used was very tiny, and confined to a very small proportion of the string's surface area.

Transfer functions were then measured, using a PC-based data-logger and a standard spectral analysis package. Various forms of input signal could have been used: for reasons of convenience and equipment availability, a fast sine sweep was employed, covering the range 0–6 kHz in one second.

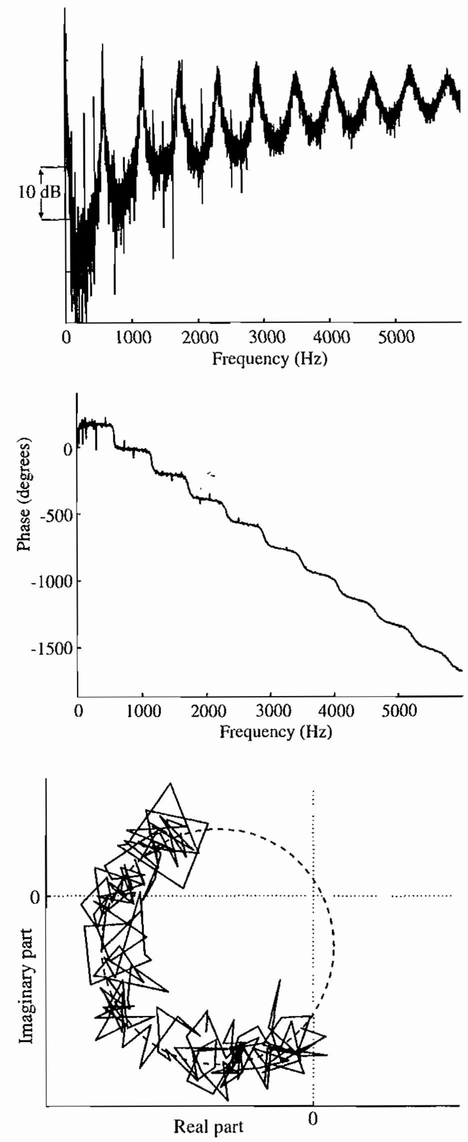

Figure 1. Torsional transfer function measured between points near the ends of a steel-cored cello D string, tensioned between rigid supports: (a) logarithmic amplitude; (b) phase; (c) Nyquist plot and best-fitted circle, for the peak near 4.6 kHz.

Input and output signals were digitised at 20 kHz, and sufficiently long FFTs were applied to both. Results for transfer torsional mobility (or admittance) such as that shown in Figure 1 were obtained. Although somewhat noisy, this shows exactly the behaviour expected. A sequence of broad torsional resonances are seen, approximately equally spaced. At low frequencies, slight traces of the narrow transverse resonances can also be seen (approximately at integer multiples of 147 Hz). Between each pair of torsional peaks is a smooth dip. There are no antiresonances because we are measuring from end to end of a one-dimensional system, and the modes are alternately symmetric and antisymmetric about the midpoint of the string. Thus the product of mode amplitudes at the driving and observing points changes sign for each successive mode, which is the condition for a smooth dip rather than an antiresonance [9]. The phase response, shown in Figure 1b, tells the same story. A steady downward drift of phase is seen, with a rounded step through 180° as each resonance is passed. (The transfer function has been smoothed by 5-point local averaging to show the phase response, since otherwise the level of noise gave unreliable behaviour from the phase-unwrapping algorithm.)

From such transfer functions it is straightforward to extract frequencies and damping factors for each peak, using a least-squares circle-fitting technique such as is commonly employed in experimental modal analysis [10]. This procedure was found to give reliable results without any prior smoothing of the transfer functions. This is illustrated by Figure 1c, which shows the Nyquist plot and fitted circle for the peak near 4.6 kHz in Figure 1a (i.e. the 8th torsional mode). The string in this case was a D-string (Astrea) constructed with a core of steel wire which carries the tension, covered with two layers of helical windings to increase the mass per unit length without increasing bending stiffness too much [6]. In this case, the inner winding was of copper wire and the outer casing of flat stainless steel. The other two strings tested had somewhat similar core+wrapping construction, but with different core materials and different wrapping details. One had a core of stranded nylon (Thomastik 'Dominant'), the other of natural gut (Kaplan 'Golden Spiral'), both having a single layer of flat metal wrapping. It had been intended that all three strings would be D strings to give direct comparability, but it emerged after the tests had been completed that the gut-cored string was in fact an A-string which had been wrongly packaged. This tiresome error muddies the results somewhat, but fortunately it does not conceal the main conclusions.

3. Results

3.1. String properties

The results for the frequencies and Q-factors for the three strings measured are listed in Table I. Several things emerge from this table. First, the frequencies are indeed approximately harmonically spaced in each case. A best-fitting fundamental frequency is easily calculated as the average value of the ratio of mode frequency to the mode number. Second, the Q factors are rather uniform for all modes of each separate string. This behaviour, which was repeated for all measurements made, allows us to summarise results for damping by giving a single averaged value for the Q factor in each case. As already mentioned, the Q factors are all very low compared with those of transverse string modes, which are typically a few hundred.

The results confirm qualitatively the earlier measurements of Gillan and Elliott [7], but in fact the damping levels from the present measurement are lower. Gillan and Elliott only measured one cello string, a steel-cored G string for which they give a damping ratio of 3.0% which corresponds to a Q value of 17. This differs from the result for the present steel-cored D string by a factor of two, as is also apparent

Table I. Measured frequencies and Q factors for the three tested strings when tuned to 147 Hz.

Mode number	Nylon freq (Hz)	Q	Gut freq (Hz)	Q	Steel freq (Hz)	Q
1	762	41	681	22	574	45
2	1504	52	1360	24	1159	41
3	2275	45	2060	20	1727	35
4	3027	55	2769	22	2314	38
5	3804	45	3464	18	2907	37
6	4542	43	4117	20	3501	30
7	5308	39	4818	17	4070	34
8	-	-	5490	20	4658	31
9	-	-	-	-	5232	31
10	-	-	-	-	5804	29

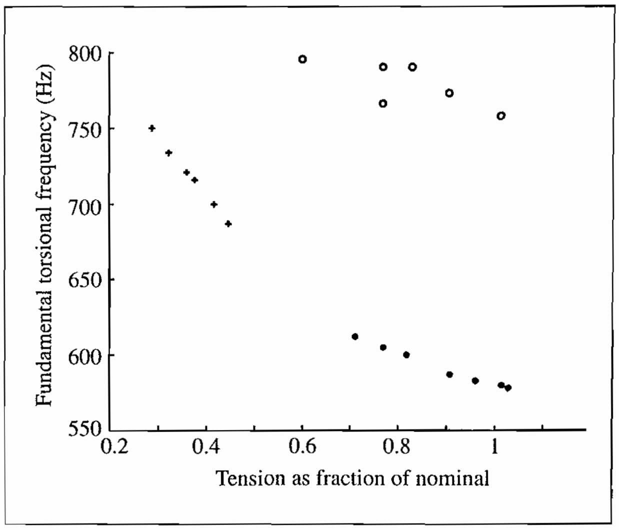

Figure 2. Best-fitted fundamental torsional frequency as a function of string tension. Open circles: nylon-cored string; solid circles: steel-cored string; crosses: gut-cored string.

by comparing their Figure 2 with the present Figure 1: the first peak-to-valley difference is about 25 dB here, whereas they show about 19 dB. This 6 dB difference fits neatly with the reported factor two in Q [9]. One can only speculate on the reason for the difference. Their string was different, and mounted on an instrument rather than the present optical table, so it may be a real effect. On the other hand, all their reported Q factors are very low indeed, as low as 6.5 for one violin D string, and it may be possible that their wire-loop drivers and transducers gave some additional damping, perhaps from the glue used to attach them to the strings. If that were so, one would expect violin strings to be affected more strongly than heavier cello strings, which is consistent with the pattern of their results.

Table II summarises the averaged torsional results for the three cello strings, along with other measured and deduced properties of the three strings. These additional properties make up the same set as discussed by Schumacher [2], and ıble II. Properties of the three tested strings when tuned to 147 Hz otation following Schumacher [2]).

ɔre material	Nylon	Gut	Steel
̇ass per unit length m (kg/m, $\cdot 10^{-3}$)	2.7	1.5	3.8
iameter $2a$ (mm)	0.97	1.08	0.93
ominal tension (N)	111	139	156
·ansverse wave speed c_T (m/s)	206	308	206
:ansverse wave impedance Z_T (kg/s)	0.55	0.46	0.77
ɔrsional fundamental frequency (Hz)	758	687	580
verage torsional Q factor	46	20	34
ɔrsional wave speed c_R (m/s)	1060	962	810
ɔrsional stiffness (pendulum) ' (Nm^2, $\cdot 10^{-4}$)	8.1	–	4.2
ɔlar moment per unit length ı (kg m, $\cdot 10^{-10}$)	4.0	3.2	4.2
elative radius of gyration k/a	0.87	0.86	0.72
ɔrsional impedance $'_R = \Theta c_R/a^2$ (kg/s)	1.8	1.1	1.6
ıferred torsional stiffness ' (Nm^2, $\cdot 10^{-4}$)	4.5	3.0	2.8

is notation is followed. They are all properties which are re- uired input for a realistic simulation of the action of bowing. 'ransverse properties (tension, wave speed and impedance) ıave been calculated from the nominal length and pitch of he strings, together with measurements of the mass per unit ength. The torsional wave speed follows directly from the requency measurements.

For two of the strings, the torsional stiffness T was mea- ured directly using a torsional pendulum. The string was uspended vertically with a suitable end weight to give it the ame tension as in the modal measurements. The working ength was again 0.7 m. The period of torsional oscillation of he mass was measured, and from this the torsional stiffness ɟf the string is readily calculated. This calculation requires he value of the moment of inertia of the suspended mass, which was determined using a "trifilar suspension" [11]. The weight is suspended from three light cords, and the frequency ɟf torsional oscillation measured. This frequency, together with the geometric configuration, suffices to determine the moment of inertia.

The values for the torsional stiffness of the nylon-cored and steel-cored strings are listed in Table II under "Torsional stiffness (pendulum)". The same values were obtained from torsional oscillation of (relatively) large amplitude (one complete revolution) and smaller amplitude (twist of about 5° in the 0.7 m length). Both amplitudes were small enough that linear behaviour would be expected. The results present a problem. The torsional wave speed is given by

$$c_R = \sqrt{T/\Theta}, \tag{1}$$

where Θ is the polar moment of inertia per unit length of the string, so that a value for Θ can now be deduced. If this is expressed in terms of a radius of gyration k via

$$\Theta = mk^2, \tag{2}$$

where m is the mass per unit length, the values of k turn out to be $0.85a$ for the steel-cored string, and $1.09a$ for the nylon-cored string. Both of these results are too high, and the second is physically impossible since the radius of gyration cannot be greater than the physical radius of the string.

As an alternative procedure, Θ can be directly estimated for each string by the approach used by Schumacher, based on measuring the thickness and density of each of the layers making up each string and using the theoretical expression for moment of inertia of concentric cylinders. These values for Θ and quantities deduced from it, listed in Table II, are physically sensible and seem quite compatible with the results given by Schumacher [2] for violin strings. In particular, the quantity $(k/a)^2$ now takes the value 0.52 for the steel-cored string, which is more or less the value for a solid string, while the two wrapped strings give values 0.76 and 0.74, close to the value 0.73 which Schumacher reports for several wrapped strings.

From these estimates of Θ an alternative estimate of the torsional stiffness can be deduced from the measured wavespeed using equation (1). These values, listed as "Inferred torsional stiffness", are conspicuously smaller than the torsional pendulum measurements. This discrepancy seems to be significantly larger than can be accounted for by the various experimental uncertainties. The only obvious physical difference lies in the fact that the torsional pendulum operates under near-static conditions (oscillation periods of a minute or two), whereas the torsional wave speed was determined at kilohertz frequencies. Could there be some frequency dependence in the behaviour of wrapped strings like these? It is far from clear how this might arise, but no other explanation suggests itself at present.

3.2. Influence of tension

The measurements of torsional mode frequencies and Q factors were repeated at a range of tensions, for each of the three strings. The results for frequencies are summarised in Figure 2. For each string, there is a clear trend of decreasing torsional frequencies with increasing tension. Tensions were not raised significantly above the original "tuned" state, for fear of breaking the strings. This figure also highlights the fact that the gut-cored string was discovered, subsequent to the tests, to have been an A string rather than a D string, so that the range of tension explored is well below the "nominal" tension. The trend with tension is still clear, though. It appears that had this string been tuned to its correct frequency, the torsional frequencies might have fallen to be comparable with those of the steel-cored string.

The explanation of the falling trend of torsional frequencies is, presumably, to do with the wrapped construction of the strings. One possible reason is simply that the string diameter decreases a little with increasing tension, reducing Θ and increasing c_R. Another is that as tension increases,

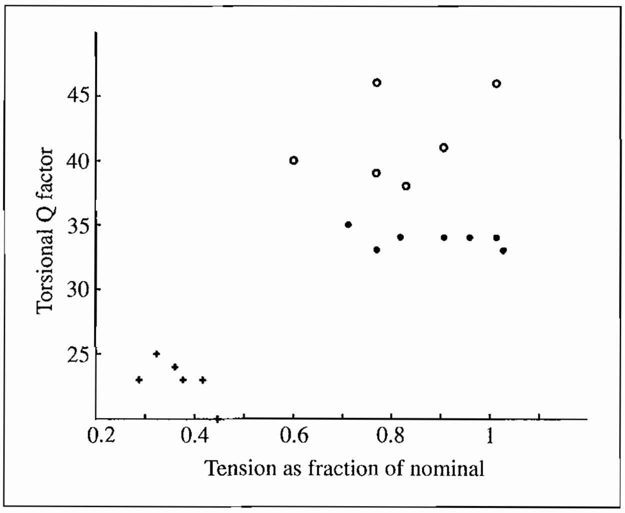

Figure 3. Average torsional Q-factor as a function of string tension. Open circles: nylon- cored string; solid circles: steel-cored string; crosses: gut-cored string.

the wrappings will tend to be pulled somewhat apart. However, strings are made with the core under tension so that in the relaxed state the wrappings are presumably under some compression. The result may be that adjacent turns of the wrappings maintain contact over the normal range of tension, and are able to transmit some torsional stress. As tension increases the shear stiffness of these contacts may reduce, thus reducing the overall torsional stiffness of the string.

In the case of the string with a stranded nylon core, it should be noted that the tension force is carried by the individual fibres. Distorting these fibres during a torsional deformation should give rise to an additional restoring moment, and thus contribute to the torsional stiffness. This contribution would tend to *increase* with tension, so if present it is certainly not the dominant effect.

Figure 3 shows the results for torsional damping as a function of tension. The gut-cored and steel-cored strings show Q factors which are unchanged within the accuracy of the measurement. The nylon-cored string shows more scatter, but this may simply reflect the greater scatter of Qs between modes for this string (see Table I). Figure 3 shows the mean value, and for this string the associated standard deviation is relatively large. The conclusion is that strings of different construction have different torsional damping, essentially independent of tension. These differences of damping may contribute to the different perceived playing properties of gut-, nylon- and steel-cored strings.

4. A reflection function for constant Q

The results described in the previous section constitute a more complete characterisation of musical strings than any previously published. To be able to use this relatively complete knowledge for the purposes of simulation of the bowing process, one further stage is needed. The most efficient simulation algorithm does not work in terms of modal properties, but instead requires the information about reflecting torsional waves to be encapsulated in "reflection functions" [1]. These are defined such that the torsional wave returning to the bowed point from one end of the string is the convolution of the outgoing wave with the reflection function. The two reflection functions, for the two sections of the string separated by the bowed point, thus encapsulate all relevant information about dispersion, reflection processes, and dissipation.

The evidence of Table I, and the other measurements made in this study, suggest that a suitable model for torsional waves on these strings involves no dispersion, and has damping which gives all modes the same Q factor. The rather high damping together with the constant-Q behaviour strongly suggests that the damping has its origin in distributed dissipation in the material of the string, rather than in losses during reflection at the ends of the string (for example due to microslipping of the string in the bridge notch). This has been confirmed by additional tests in which the boundary conditions were modified by gluing the string to the supporting "bridges". This made no measurable difference to the damping.

A suitable reflection function to describe this idealised behaviour is easily derived. The starting point is to consider the evolution of a delta-function pulse as it travels along a one-dimensional system subject to ideal "hysteretic" damping. As is well known, idealised hysteretic behaviour is not physically possible because it violates causality [12]. This fact will be ignored initially, but it will soon be seen that a causal approximation which is adequate for the present purpose can be obtained. In the absence of dissipation an ideal pulse travelling in the x-direction will be simply $\delta(t - x/c_R)$. This can be decomposed into frequency components via

$$\begin{aligned}\delta(t - x/c_R) &= \frac{1}{\pi}\int_0^\infty \cos\omega(t - x/c_R)\,\mathrm{d}\omega \\ &= \frac{1}{\pi}\Re e \int_0^\infty \mathrm{e}^{\mathrm{i}\omega(t-x/c_R)}\,\mathrm{d}\omega. \end{aligned} \quad (3)$$

In the presence of hysteretic damping, each wave component would decay as it propagates so the required travelling pulse becomes

$$h(t,x) = \frac{1}{\pi}\Re e \int_0^\infty \mathrm{e}^{\mathrm{i}\omega(t-x/c_R)-\omega\varepsilon x/c_R}\,\mathrm{d}\omega, \quad (4)$$

where ε is a (small) constant equal to $1/2Q$. The integral may be evaluated straightforwardly to give

$$h(t,x) = \frac{\varepsilon x/c_R}{\pi\left[(t - x/c_R)^2 + (\varepsilon x/c_R)^2\right]} \quad (5)$$

and the required reflection function is simply the negative of this. Note that

$$\int_{-\infty}^{+\infty} h(t,x)\,\mathrm{d}t = 1 \quad (6)$$

for any fixed x, a necessary condition on any physically-plausible reflection function [1].

If this is thought of as the evolving shape, for $t > 0$, of a delta function launched at $t = 0$, $x = 0$ then it suffers

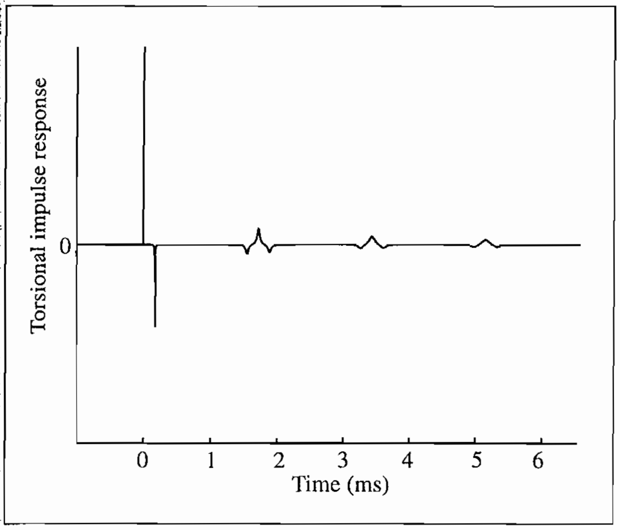

Figure 4. The first few cycles of a torsional impulse response function simulated using the constant-Q reflection function derived in the text, with parameter values as measured on the steel-cored string. Driving and observing points are both 1/10 of the string length from one end.

Table III. Frequencies and Q factors from the simulated torsional impulse response shown in Figure 3, to be compared with the results for the steel-cored string in Table I.

Mode number	Frequency (Hz)	Q factor
1	581	39.0
2	1162	36.8
3	1744	35.9
4	2325	35.3
5	2906	35.0
6	3488	34.7
7	4069	34.6
8	4650	34.4

from the obvious disadvantage of being non-causal. At a fixed position with non-zero x, the "tail" of the pulse is non-zero for negative time as well as for positive time. This is the manifestation in this particular problem of the well-known non-physical behaviour of ideal hysteretic damping [12]. However, for the pragmatic purposes of simulation a sufficiently good approximation is obtained by making a causal version of this function by simple truncation. This approximation is only likely to work well, of course, when the damping is sufficiently light that the temporal spread of the pulse is relatively small compared with the time delay to the pulse centre. This requires $\varepsilon \ll 1$, or equivalently $Q \gg 1$. This condition is adequately met by the Q values measured in the tests reported here.

To test this idea, a pair of reflection functions calculated in this way can be used to reconstruct the impulse response at any given point on the string. This can then be analysed as if it were experimental data, to determine the modal frequencies and Q factors. As an example, consider a model of the steel-cored string driven and observed at a point 1/10 of the string length from one end. The required value of ε is $1/(2 \times 34) = 0.015$ to match the observed Q factor. The initial part of the computed impulse response is shown in Figure 4. After the initial delta-function response at $t = 0$, the first thing seen is the inverted reflection from the end of the string closest to the measurement point. The pulse has not propagated far, so the pulse is still quite narrow – the pulse amplitude predicted by equation (5) for negative times, ignored to make the signal causal, was very small indeed. By the time the reflection from the other end of the string arrives, it has spread further, then the two travelling pulse return simultaneously after one complete round trip, to produce a positive-going pulse. The pattern repeats in subsequent cycles, with the pulses steadily spreading.

When this function is analysed, the first few mode frequencies and Q factors are as listed in Table III. The frequencies are very close to integer multiples of 581 Hz, as expected. The Q factors are all rather close to the expected value (34). The lowest modes have slightly higher values – this is the expected result of truncation of the reflection function, since the lowest frequencies are most affected by the tails of the spreading pulses. These results suggest that this simple model does indeed allow the observed torsional behaviour to be captured with sufficient accuracy to be used in simulations of bowing.

5. Conclusions

The torsional behaviour of three different cello strings has been measured, using a novel optical sensor and a wire-loop drive system such as has been used previously [6, 7]. The strings were mounted between rather rigid bridges fixed to a heavy optical table. Torsional frequencies were measured up to 6 kHz, and the corresponding Q factors were also determined. It was found that for all types of string the resonant frequencies were harmonically spaced to reasonable accuracy, and that the Q factors were similar for all modes. However, the torsional wave speed, impedance and damping varied between the different types of string. The lowest damping was found in a string with a stranded nylon core and metal over-wrapping. A steel-cored string had higher damping, and a gut-cored string the highest of all. Although the strings measured were all different from those tested by Gillan and Elliott [7], it seems that significantly lower damping was found in this test than they reported.

An independent measurement of torsional stiffness of two of the strings was made, using a torsional pendulum operating at very low frequency. This gave results which were not entirely consistent with the wave-speed determination. The reason for this discrepancy is not clear, but it is possible that torsional behaviour in these strings is frequency-dependent, perhaps connected with their multi-layer wrapped construction. This construction may also be responsible for the observed effects of string tension: it was found that the torsional wave speed decreased with increasing tension in all three strings. The damping was unaffected by tension.

Finally, these results have been expressed in a form suitable for use in simulations of the bowing process. For this purpose, the properties of propagating and reflecting torsional waves must be described via "reflection functions" [1], the kernel functions with which outgoing torsional waves must be convolved to give the incoming reflected waves. A novel reflection function has been presented which reproduces the non-dispersive "constant-Q" behaviour seen in the torsional measurements. Although the idealised form of this function is non-causal, it has been demonstrated that a version made causal by truncation reproduces the observed behaviour quite well. This will allow simulation studies to be made, to test the significance of the torsional behaviour revealed here for the "playability" of musical strings on a bowed instrument.

Acknowledgement

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