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DYNAMIC EFFECT OF SLAB DISCONTINUITY ON UNDERGROUND MOVING TRAINS

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Abstract

Floating slab track is commonly used as a mean of vibration isolation in underground tunnels. This involves mounting the entire track on a concrete foundation slab that rests on rubber bearings or steel springs. The slab may be cast in-situ, resulting in a continuous length of concrete, or may be constructed in discrete pre-cast sections laid end to end. In the latter case, dynamic loads are induced at the wheel-rail interface due to slab discontinuity. This paper discusses a new method for modelling tracks with discontinuous slabs under moving trains. Rails and slab are modelled as Euler-Bernoulli beams. The modelling is carried out in the frequency domain for a unit with a prescribed slab length, where the periodic infinite structure theory is used to account for periodicity in the longitudinal direction. The response of the track due to oscillating moving loads with different forward speeds is investigated. These results are compared with ones based on a stationary-load solution. Fourier series is used to couple a finite model to represent a moving train on the track using results of track's response to oscillating moving loads. This gives a converged solution by using only a limited number of harmonics, which enables the dynamic effect of discontinuous slabs to be investigated.

INTRODUCTION

Vibration generated by trains has a great environmental impact on buildings near underground tunnels. Many mechanisms generate dynamic forces at the wheel-rail interface. Examples are wheel-rail roughness, sleeper spacing, and rail joints.

Vibration at the wheel-rail interface is transmitted to the tunnel wall and propagates to nearby buildings. This vibration is perceived by inhabitants either directly as vibration in floors and walls or indirectly as reradiated noise. A third and very significant source of disturbance is due to movement of household objects, especially mirrors, windowpanes and glassware that rattles in the cupboard.

Vibration isolation can be successfully achieved by using some countermeasures at the source. These include using soft rail-pads, resiliently mounted sleepers, low-stiffness vehicle suspension, and a floating slab track. Floating slab track is widely known as an effective measure of vibration isolation and it is used at many underground tunnels, see for example G.P. Wilson et. al. [7]. The railway track is mounted on a concrete slab that rests on rubber bearings or steel springs. The slab can be continuous if it is casted in-situ or discontinuous if it is constructed in discrete pre-cast sections. For discontinuous slabs, additional dynamic loads are induced at the wheel-rail interface.

This paper discusses modelling and analyzing discontinuous floating slabs under moving trains. A moving wheel on the track experiences a change of stiffness due to slab discontinuity. The stiffness under the moving wheel is a periodic function in space with periodicity equal to the slab length. The change in stiffness excites the train's inertial forces, which in turn increases the input on the rail. For heavy axle masses and high-speed trains, significant dynamic forces are induced at the wheel-rail interface.

The tunnel wall is modelled in this work as a rigid foundation. This assumption is appropriate when studying dynamic loads at the wheel-rail interface. However, a detailed model of the tunnel and the ground is needed when studying ground-borne vibration [3]. To calculate the track response under a moving train, calculations of track response under a moving oscillating load should be performed first. The track is modelled as a periodic infinite structure and can be analyzed by one of the following two methods:

1. Floquet-Fourier method: two steps are used to calculate the response of a periodic infinite structure under a harmonic-moving load. First, the structure response under a stationary harmonic load is calculated using Floquet's theorem. The second step is to integrate the result weighted by the frequency-domain force to calculate the frequency domain response. Then by using Fourier transform, the time domain response is calculated. Nordborg [4,5] uses this method to model a rail periodically supported on discrete supports. He calculates a closed-form solution for rail displacement under a non-moving oscillating load. This produces a quick model to calculate the rail response under oscillating moving loads. Forrest [2] calculates the response of floating track with discontinuous slab under stationary oscillating loads using the stiffness and the repeating unit methods. Displacements are only calculated for a harmonic concentrated force applied on the rail above a slab discontinuity.

2. Periodic-Fourier method: in this method calculations are made only for one repeating unit. Responses of other units are calculated using the periodicity condition. Equations of motion of the unit under consideration are transformed to the frequency domain. The resulting differential equations are written as a summation of homogenous and particular solutions. The homogenous solution coefficients are found by considering the boundary condition at ends of the unit under consideration. These results are then transformed back to the time domain. This method is used by Smith and Wormley [6] for non-oscillating moving loads applied on periodically supported beam. Belotserkovskiy [1] uses this method to analyze a rail on uniform foundation with periodically resilient hinges to represent rail joints and with discrete supports to account for sleepers, under a harmonic-moving load. This method is more direct and will be described briefly in this paper.

OUTLINE OF THE MODEL

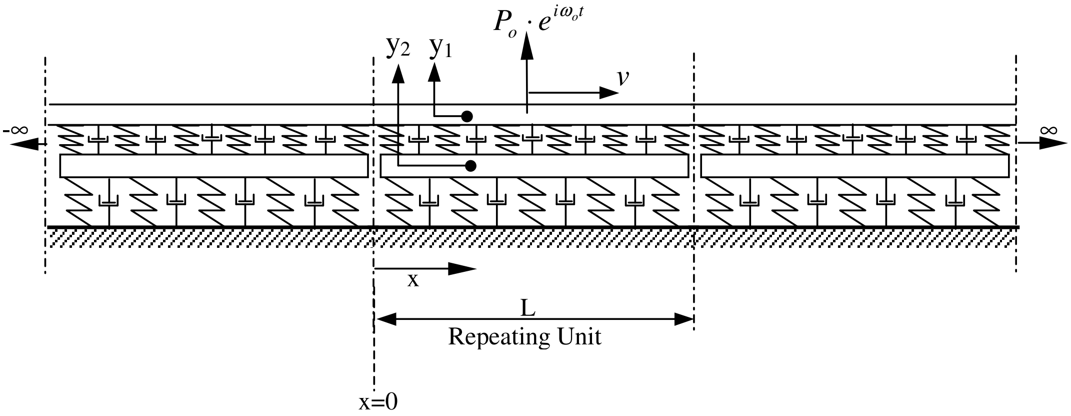


Figure 1: Floating track with discontinuous slab subjected to a moving harmonic load.

The model is shown in Figure 1 and consists of an upper Euler-Bernouli beam to account for both rails (with mass m_1 per unit length and bending stiffness EI_1) and a lower Euler-Bernouli beam to account for floating slab (with mass m_2 per unit length and bending stiffness EI_2). The slab length is L. Rail pads are represented by a continuous layer of springs with stiffness k_1 and viscous damping with damping factor c_1. Slab bearings are represented by a continuous layer of springs with stiffness k_2 and viscous damping with damping factor c_2. The rail is loaded by a harmonic moving load of magnitude P_o, angular frequency ω_o, and velocity v. Responses of rails and slab are $y_1(x,t)$ and $y_2(x,t)$ respectively.

PERIODIC-FOURIER METHOD

For a periodic structure such as the one in Figure 1, one can write the following periodicity condition:

$$y(x+L,t+\frac{L}{v}) = e^{i\omega_o L/v} \cdot y(x,t) \tag{1}$$

This is because point 1 with coordinates (x,t) and point 2 with coordinates $(x+L, t+L/v)$, have similar loading conditions (note that t determines the load's phase, and its position along the track). The distance between the moving load and the measuring point is the same and the load frequency is the same. The only difference is that the load magnitude for point 2 is $e^{i\omega_o L/v}$ times the load for point 1.

At any point x for the repeating unit defined between $x = 0$ to $x = L$ in Figure (1), and at any time t, one can write two differential equations of the forth order for beams (equations (2) and (3)). Also 8 boundary conditions can be written at ends of the repeating unit (equations (4) and (5)). The boundary-conditions equations are derived from the periodicity condition and continuity of displacements, slopes, moments, and shear forces at ends of the repeating unit.

$$EI_1 \frac{\partial^4 y_1}{\partial x^4} + m_1 \frac{\partial^2 y_1}{\partial t^2} + k_1(y_1 - y_2) + c_1(\frac{\partial y_1}{\partial t} - \frac{\partial y_2}{\partial t}) = P_o \cdot e^{i\omega_o t}\delta(x - vt) \tag{2}$$

$$EI_2 \frac{\partial^4 y_2}{\partial x^4} + m_2 \frac{\partial^2 y_2}{\partial t^2} + k_2 y_2 - k_1(y_1 - y_2) + c_2 \frac{\partial y_2}{\partial t} - c_1(\frac{\partial y_1}{\partial t} - \frac{\partial y_2}{\partial t}) = 0 \tag{3}$$

$$\frac{\partial^n y_1}{\partial x^n}(L, t+\frac{L}{v}) = e^{i\phi_o} \cdot \frac{\partial^n y_1}{\partial x^n}(0,t) \qquad \text{n=0,1,2,3} \tag{4}$$

$$\frac{\partial^m y_2}{\partial x^m}(L, t+\frac{L}{v}) = \frac{\partial^m y_2}{\partial x^m}(0,t) = 0 \qquad \text{m=2,3} \tag{5}$$

Where: $\phi_o = \frac{\omega_o L}{v}$ is the non-dimensional loading frequency.

These equations can not be solved directly in this domain due to existence of delta function in equation 2. For any x, these equations are transformed to the frequency domain and the differential equations are solved to give the responses y_1, and y_2. These results are then transformed back to the time domain.

Figure 2 shows the rail response under a non-oscillating moving load for a slab length L=6m and velocities 1 km/hr, 80 km/hr, 150 km/hr and 300 km/hr. The response of the track is the same for velocities up to 80km/hr. For underground trains where speed is restricted to velocities (usually maximum of 60km/hr) to allow for train stopping at stations, the quasi-static solution (as confirmed by Figure 2) is sufficient. Due to varying stiffness under a slowly moving load, a parametric excitation occurs for a constant moving load due to discontinuous slabs.

An approximate method to produce track's response under moving loads is achieved by modifying the track response under stationary loads (computed by Forrest's model [2]). Figure 3 explains the procedure to do this. Figure 3.a shows an oscillating stationary load, which stands at $x = x_o$. The response underneath this load

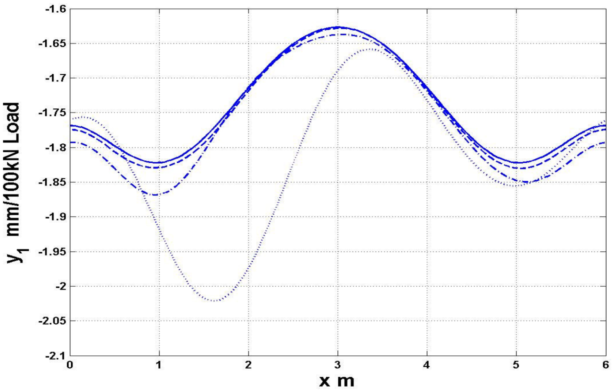
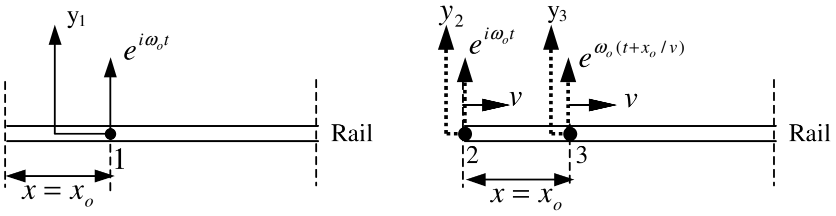


Figure 2: Rail displacement under a downward static moving load with velocities: 1 km/hr (solid), 80 km/hr (dash), 150 km/hr (dash-dot) and 300 km/hr (dot). Parameters used in the analysis are EI_1*=10MPa/m*^{2}*,* m_1*=100kg/m,* k_1*=40MN/m/m,* c_1*=20kN/m/(m/s),* EI_2*=1430MPa/m*^{2}*,* m_2*=3500kg/m,* k_2*=50MN/m/m,* c_2*=50kN/m/(m/s)*

is $y_1 = Y_1 e^{i\omega_o t}$. In Figure 3.b, an oscillating force starts at x=0 with phase equal to zero (see equation 2). This load moves (very slowly), until it gets to $x = x_o$. At this point the force will be $F = e^{i\omega_o (t + x_o / v)}$. By comparing with the non-moving load case, the response underneath the load will be $y_3 = Y_1 e^{i\omega_o t} e^{i\omega_o x_o / v}$. Hence to compute the moving load displacement, the non-moving load displacement is modified by the factor $e^{i\omega_o x_o / v}$.



(a) (b)

Figure 3: Calculating moving-load results using Forrest model. (a) Forrest model, response is measured under an oscillating non-moving load at point 1. (b) Current model, moving load starts at point 2 where x=0 with zero phase and gets to point 3 where $x = x_o$ *with phase* $\omega_o x_o / v$*.*

Figure 4 (a, b, c, and d) compares the rail displacement computed by the genuine moving load model and by the phase-modification-stationary-load model. Matching is observed at a, c, and d. Some difference can be noticed at (c) where the loading frequency matches with a cut-on frequency of the track.

For a harmonic-moving load with frequency equal to a multiple of the slab-passing frequency in the form: $R = e^{i\omega_n t}$, where $(\omega_n = \frac{2\pi n v}{L})$, the rail response can be written as a Fourier series sum as follows:

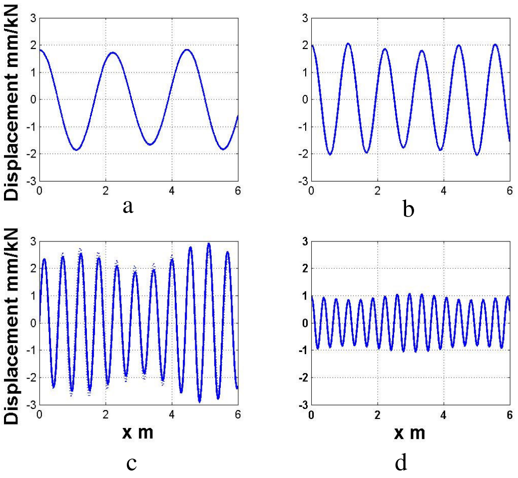


Figure 4: Rail displacement under a moving load computed by the Periodic-Fourier method (solid), and by phase modification of a stationary load results using Forrest's model (dot) for 40km/hr moving load with oscillating frequencies: (a) 5 Hz, (b) 10 Hz, (c) 20Hz and (d) 30 Hz. Parameters are given in the caption of Figure (2).

$$y_r = \sum_{q=-p}^{+p} h_{q,n} e^{i\omega_q t} \tag{6}$$

Where: $p \to \infty$, $h_{q,n}$ is the magnitude of the qth harmonic of the rail displacement when a unit oscillating moving load is applied to the rail with angular frequency ω_n. This relationship is used to couple a train model to the track.

COUPLING A TRAIN MODEL

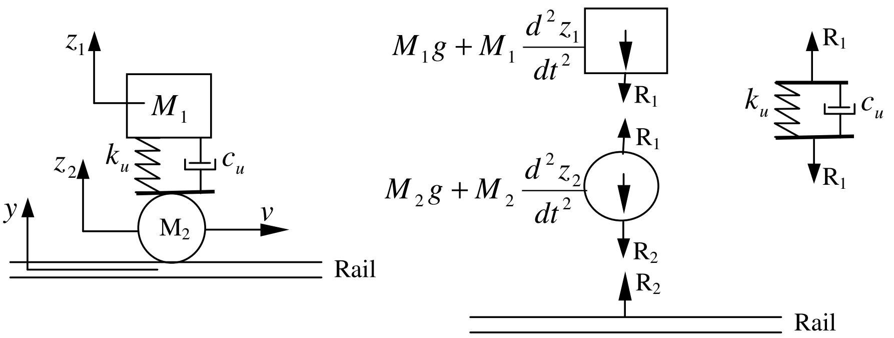
The train model and its free-body diagrams are shown in Figure 5. z_1, z_2, R_1, R_2, and y are all periodical functions of time. They give the train's displacements, forces, and the rail displacement respectively. Note that rail and wheel are assumed smooth in this analysis.

It is possible to write train's displacement as a sum of Fourier series. The unsprung mass, sprung mass displacement can be written as:

$$z_1 = \sum_{n=-S}^{+S} G_n \cdot e^{i\omega_n t} \qquad z_2 = \sum_{n=-S}^{+S} C_n \cdot e^{i\omega_n t} \qquad \text{with } (\omega_n = \frac{2\pi n v}{L}) \tag{7,8}$$

Where S should be large enough to capture all the significant harmonics. For equilibrium of the unsprung mass and the sprung mass respectively:

$$R_1 = -M_1 g - M_1 \frac{\partial^2 z_1}{\partial t^2} \qquad R_2 = R_1 - M_2 g - M_2 \frac{\partial^2 z_2}{\partial t^2} \tag{9,10}$$



(a) (b)

Figure 5: (a) Coupling a model of a half vehicle to the track. (b) Free-body diagrams. Only the rail is shown from the track model.

The force-displacement equation of the suspension reads:

$$R_1 = k_u (z_1 - z_2) + c_u \left(\frac{\partial z_1}{\partial t} - \frac{\partial z_2}{\partial t}\right) \tag{11}$$

Solving for R_2 from equations 9 and 10, using equations 7 and 8 gives:

$$R_2 = -(M_1 + M_2) g + \sum_{n=-s}^{+s} [M_1 G_n \omega_n^2 + M_2 C_n \omega_n^2] \cdot e^{i\omega_n t} \tag{12}$$

The rail displacement due to R_2, can be written as a Fourier series summation of terms $h_{q,p}$ as explained in the previous section. As this displacement is equal to the unsprung mass displacement in equation 8. The harmonics of these two expressions are equated and the coefficients in equation 7 and 8 are calculated.

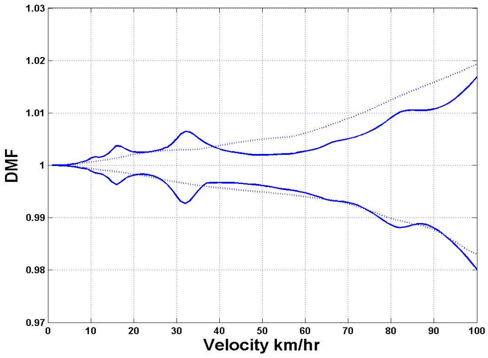


Figure 6: Dynamic magnification factor. This gives maximum and minimum forces at the wheel-rail interface. These forces are normalised by dividing by $(M_1 + M_2)g$. Solid and dotted curves are for train suspension damping ratio equal 0.1, and 0.5 respectively. L=6m,M$_1$=7000kg,M$_2$=4000kg, ku=1500kN/m Other parameters are given in the caption of Figure (2).

Figure 6 shows the DMF for the rail-wheel forces. Resonance occurs when the natural frequency of the sprung mass on the stiffness of the primary suspension or the unsprung mass on the track matches with the slab-passing frequency.

CONCLUSIONS

This paper presented a method to analyse floating track with discontinuous slabs under moving trains. Slab discontinuity is shown to provide a parametric excitation for a moving load over the track. This is of significant importance especially for high-speed moving trains as well as for typical trains with heavy unsprung masses. For low velocity oscillating loads, it was shown how to use results of track response for a stationary oscillating load to produce results of track response for a moving oscillating load by using the phase information of the input.

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