An approximate method for the solution of an influence function foil rolling model

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Abstract

Fleck and Johnson (Int. J. Mech. Sci. 29 (1987) 507) and Fleck et al. (Proc. Inst. Mech. Eng. 206 (1992) 119) have developed foil rolling models which allow for large deformations in the roll profile, including the possibility that the rolls flatten completely. However, these models require computationally expensive iterative solution techniques. A new approach to the approximate solution of the Fleck et al. (1992) Influence Function Model has been developed using both analytic and approximation techniques. The numerical difficulties arising from solving an integral equation in the flattened region have been reduced by applying an Inverse Hilbert Transform to get an analytic expression for the pressure. The method described in this paper is applicable to cases where there is or there is not a flat region.

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1. Introduction

The estimation of roll pressure, roll torque, contact length and forward slip are important in the design and operation of rolling mills. Until recently, rolling mills were set-up primarily using experience obtained on similar mills or on experimentation on the new mill, although some
guidelines could be obtained from the existing mathematical models of the rolling process. This experimentation was both time consuming and expensive. As the models have improved, they have been used to assist in the setting up process either by using lookup tables from off-line simulation programs or on-line set-up models. However, there is a need for accurate mathematical models coupled with rapid numerical solution methods.

The earliest work on cold rolling was conducted by von Kármán [1], Siebel [2] and Orowan [3]. In their research, they used the simplifying assumption that the rolls remained circular in profile. The earlier models, such those by Orowan [3] and Bland and Ford [4], had a great deal of success in predicting the stresses in cold rolling of thick strip. These models were based on a circular arc of contact and use the Hitchcock formula [5] to predict a deformed roll radius. When these earlier

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**Nomenclature**

- $a$: contact width from the centre-line of the rolls to entry or exit location (mm)
- $\sigma_x$: horizontal stress (MPa)
- $b$: semi-thickness of the strip (mm)
- $p$: pressure (MPa)
- $q$: shear stress (MPa)
- $Y_s$: yield stress (MPa)
- $R$: roll radius (mm)
- $x_i$: positions of zone boundaries as shown in Fig. 1 (mm)
- $w_i$: scaled positions of the zone boundaries (dimensionless)
- $E^*_R$, $E^*_S$: plain strain Young’s modulus of the roll and strip, respectively (GPa)
- $\mu$: coefficient of Coulomb friction (dimensionless)
- $v_{rS}$: Poisson’s ratio of the strip and roll material, respectively (dimensionless)
- $\Delta x$: grid spacing in the numerical method (mm)
- $\alpha$: direction of friction $+1$ ($-1$) forward (backward) slip
- $-x^*_0, x^*_2$: dimensionless entry and exit coordinate positions (dimensionless)
- $\varepsilon$, $U_0$, $\eta_0$, $\tau$, $\kappa$: dimensionless constants (dimensionless)

**Subscripts**

- $S$: strip
- $R$: roll
- 0: quantity at the entry of the roll gap
- 1: quantity at the central flattened zone
- 2: quantity at the exit of the roll gap

**Superscripts**

- *: non-dimensionalised terms
- #: quantity at the reference position in the elastic regions
models were applied to the rolling of thin foil, where the roll is far from circular, convergence problems arose. Fleck and Johnson [6] and Fleck et al. [7] departed from the traditional assumption that the rolls remained circular. Instead they assumed that, when the rolls deformed, the roll profile included a central flattened region. In Fleck and Johnson model the pressure was taken to be near Hertzian. The deviation of the roll profile from flat was incorporated by using a modified Winkler mattress model. Work by Ståhlberg and Keife [8] also assumed the pressure was a modified Hertzian but in their approach the deviation was taken from the inclined line joining the entry and exit thicknesses. Recently the work of Shi, McElwain and Langlands [9] and Langlands and McElwain [10] have given an approximate solution of the Fleck and Johnson modified Hertzian model.

Jortner et al. [11] also assume the roll profile is non-circular. However, here they use an influence function based on the deformation of an elastic cylinder. Grimble et al. [12] used Jortner’s influence function in combination with Golten’s shear influence function. Jortner’s influence function has been used by Domanti and Edwards [13] to develop their temper rolling model.

Sutcliffe and Rayner [14] have shown experimentally that a central flattened region may occur in foil rolling. The experiment involved the rolling of plasticine strips. The process was stopped and the rolls were removed. Their results also showed there was a degree of strip thickening in the central flattened region for the thinnest cases tested. Keife et al. [15] propose that the strip undergoes plastic expansion. This expansion is thought to occur either during the rolling process and/or during the subsequent forging stage, when the rolls have stopped rotating. Le and Sutcliffe [16] removed the assumption that the strip was absolutely flat in the central region, instead using Hooke’s law, modified for plane strain, to calculate the pressure profile there. The strip profile was still predicted to be remarkably flat in that region.

There have also been numerous approaches to modelling of the rolling of foil including the use of finite element methods [17,18]. These models give more realistic representations of the geometry/physics of the rolling process incorporating lubrication, thermal and speed effects [17,19]. However, these models are not suitable for on-line simulations as they are computationally expensive.

In the following section a brief description of the Fleck et al. model and outline of the approximate method of its solution is given. The approach in this paper takes advantage of the solution of the Aerofoil Integral Equation. The approximate model’s results are then compared with the full rolling model’s predictions. It should be noted here that the main focus of this paper was to develop an approximate approach to solving the Fleck et al. Influence function model which reduces the amount of computation (such as in strip profile evaluation and the flat region pressure solution). More realistic models do exist and the approach of this paper can be easily adapted to include aspects of these models including yield-stress variation.

1.1. Influence function model

In the Influence Function Model of Fleck et al. [7] it is assumed that there are a number of zones with the actual number depending on the regime of rolling behaviour. These zones in order are: a Plastic Reduction zone with backward slip, B; a central flattened region, C–E; another Plastic Reduction zone with backward slip, F; and a Plastic Reduction zone with forward slip, G. The central flattened region can be further split into a contained plastic zone with no-slip, C; an elastic no-slip zone, D; and another no-slip contained plastic zone, E. The elastic compression and
recovery regions at the start and end of the roll gap are neglected by Fleck et al. who consider these zones to be short. However, the pressure should drop to zero at the ends of the roll gap otherwise discontinuities in the slope of the strip profile will occur at these points for this particular model. This problem was circumvented by Fleck et al. in their numerical approach (see Section 2) by using tent functions. The approximate method developed in our work includes the elastic compression region at the entry of the roll gap, A; and the elastic recovery region at the exit of the roll gap, H. A schematic of the positions of these zones is given Fig. 1.

In each of these zones, the equilibrium equation,

$$b \frac{d \sigma_x}{dx} + (\sigma_x + p) \frac{db}{dx} + q = 0 \quad (1)$$

is applied. Note in this model the strip (foil) thickness is considered to be small compared to the length of the roll gap and its transverse width. As such it was assumed by Fleck et al. that there was no variation of stress in vertical direction. The above Eq. (1) resulted from the use of this assumption and by considering a balance of the forces acting horizontally. Note however, the stress varies as the strip moves along the length of the roll gap.

In addition to the equilibrium equation (1) the Tresca yield criterion is used in the plastic zones:

$$\sigma_x + p = Y_s. \quad (2)$$

In the approximate method, Hooke’s Law is used in the elastic compression and recovery regions:

$$\frac{b - b^#}{b^#} = \frac{1 - v_s^2}{E_s} \left( p + \frac{v_s}{1 - v_s} (\sigma_x - \sigma_x^#) \right). \quad (3)$$

where $v_s$ and $E_s$ are Poisson’s Ratio and Young’s Modulus for the strip, respectively. In the elastic compression zone, $b^#$ is taken as the initial strip thickness, $b_0$, and $\sigma_x^#$ is taken as the entry tension $\sigma_0$. Similarly, in the elastic recovery zone, $b^#$ is taken as the final strip thickness, $b_2$, and $\sigma_x^#$ is taken to be the exit tension, $\sigma_2$.

In the non-flat regions, the shear stress and the pressure are related by Coulomb’s Law of Friction

$$q = \pm \mu p \quad (4)$$

with the relevant sign (+ on entry and − on exit).
The shape of the rolls is approximated by assuming they are elastic half-spaces. This is a good approximation provided that the length of the contact zone is small compared to the radius of the rolls. This condition is satisfied in foil rolling. Fleck et al. express the roll profile as

\[ b(x) = b_0 - \frac{a_0^2 - x^2}{2R} + \frac{2}{\pi E_R^*} \int_{-a_0}^{a_0} p(x') \ln \left| \frac{a_0 + x'}{x - x'} \right| \, dx', \tag{5} \]

where \( a_0 \) and \( a_2 \) are the distances from the centre-line of the rolls to the entry and exit point of the roll bite, respectively. The first two terms on the right side of the equation form the quadratic approximation to the shape of the rolls, where \( b_0 \) is the initial strip semi-thickness and \( R \) is the radius of the undeformed rolls. The final term gives the deviation the roll surface due to the pressure where \( E_R^* \) is the plane strain Young’s modulus of the roll material.

In each of the zones there are four unknowns to be determined. These unknowns are: the strip semi-thickness, \( b \), the horizontal stress, \( \sigma_x \), the pressure, \( p \), and the frictional drag, \( q \). Depending on the zone, some of these variables may be known, in which case the solution for the remaining unknown variables is required.

Fleck et al. considered three types of rolling behaviour (see Fig. 2).

In Regime I, there are two plastic reduction zones only (one of forward and one of backward slip B and G, respectively). This regime gives a friction-hill type pressure distribution similar to that seen in results from the earlier circular models. The other two regimes include a central flattened region (zones C–E) in the model’s formulation. In Regime II the strip in the central flattened region remains plastic and consists solely of the contained plastic zone C. In Regime III zones C–E are included and the strip in the central flattened region goes from a contained plastic zone to an elastic no-slip zone (characterised by a maxima in the pressure in central flattened

![Fig. 2. Schematic of the regimes of rolling behaviour considered by Fleck et al. [7]. The representatives of the pressure (left) and strip (right) profiles are given for each regime.](image-url)
region) and then to a contained elastic slip zone. Fleck and Johnson [6] argue that the strip iselastic here as the contained plastic zone cannot extend beyond the origin as the strip would have toexperience compressive plastic strains. This violates the plastic flow rule. The elastic slip zone, E, then begins when the traction, \( q \), reaches maximal friction, \( q = -\mu p \). Sutcliffe [16] has used theelastic no-slip approximation to calculate the pressure in the flat zone. In Regimes II and III the thirdplastic reduction zone (F) is included. Our approximate model includes elastic regions, A and H, at the ends of the roll gap. Note these zones where neglected by Fleck et al. [7] due to therelative short length.

2. Solution process

The approach taken by Fleck et al. to solving these equations is to first discretise the roll gap andassume that the pressure distribution in each interval is a tent function. This, in effect, assumes that the overall pressure distribution is piecewise linear. The influence function integral can now be expressed in terms of the pressure values at the grid points. The strip profile can thenbe found by the matrix multiplication of the form \( b = Ap + b_u \), where the influence coefficientmatrix \( A = D - D_0 \) is the constant for a fixed mesh. Here \( p \) and \( b_u \) are the vectors of pressurevalues and undeformed strip values at the grid points. That is,

\[
b(x_i) = b_0 - \frac{a_0^2 - x_i^2}{2R} + \frac{2}{
\pi E_s} \sum_{j=1}^{N} (d_{ij} - d_{0j})p_j,
\]

where

\[
d_{ij} = \Delta x{(k + 1)^2 \ln(k + 1)^2 + (k - 1)^2 \ln(k - 1)^2 - 2k^2 \ln k^2}
\]

as given in Fleck et al. [7] and \( k = i - j \).

In addition, Fleck et al. establish an updating formula for the pressure in the plastic reduction regions. This was achieved by dividing Eq. (1) by the strip semi-thickness, \( b \) and integrating the resulting equation using the further assumption that the reciprocal of the strip profile was piecewise linear in addition to the piecewise linear pressure profile assumption. The resulting updating formula is

\[
p_i = \frac{Y_s \ln(b_i/b_{i-1}) + p_{i-1}(1 + \rho \Delta x(\frac{1}{3b_{i-1}} + \frac{1}{6b_{i-1}}))}{1 - \rho \Delta x(\frac{1}{3b_i} + \frac{1}{6b_{i-1}})}.
\]

To solve for the pressure in the central flattened region the integral equation \( b(x) = b_1 \) needs to be satisfied. After discretisation, this gives a system of linear equations to be solved.

\[
b_1 - b_0 + \frac{a_0^2 - x_i^2}{2R} - \frac{2}{
\pi E_s} \sum_{x_i \in flat} (d_{ij} - d_{0j})p_j = \sum_{x_i \in flat} (d_{ij} - d_{0j})p_j.
\]

The pressure and strip profiles are updated iteratively. The estimated positions of the zone boundaries are then updated.
3. Approximate method

3.1. Motivation

Here, we develop an alternative approach to the solution procedure based on the observation that the integral equation in the flattened region can be solved analytically using an Inverse Hilbert Transform.

The assumptions of constant yield stress as in the Influence function model of Fleck et al. is maintained in this approach. It should be noted foil rolling will entail a number of passes at fairly high reductions. Intermediate anneals are used to soften the material during this process. In the first pass after an immediate anneal, the strip will undergo significant strain hardening which may not be adequately represented by a constant yield stress. After this the yield stress continues to grow with reduction but this effect is not significant and a constant yield stress model would be acceptable for these passes. It also worth noting the yield stress change due to hardening probably does not occur in the flattened region, so the errors are confined to the regions of plastic reduction regions at the entry and exit. It should be noted that with the inclusion of strain hardening the approach below can be modified to take the variable yield stress into account.

3.2. Nondimensionalisation

The main variables of the Influence Function model are non-dimensionalised as follows:

$$
\sigma^*_x = \frac{\sigma_x}{Y_s}, \quad x^* = \frac{x E_R}{R Y_s}, \quad p^* = \frac{p}{Y_s}, \quad \tau = \frac{E_R^*}{E_s^*},
$$

$$
q^* = \frac{q}{\mu Y_s}, \quad \varepsilon = \frac{Y_s}{E_R}, \quad b^* = \frac{b}{b_0}, \quad \kappa = \frac{v_s}{1 - v_s},
$$

where $\tau$ is the ratio of Young’s modulus of the rolls to the strip, $\varepsilon$ is the small dimensionless ratio of the yield stress of the strip to the modulus of the rolls and $\kappa$ is the dimensionless constant influencing the strip thickness deflection due to the horizontal stress in the elastic compression and recovery zones.

The main governing of the model can then be written in terms of these variables. The equilibrium equation (1) becomes

$$
b^*_x \frac{d\sigma^*_x}{dx^*} + (\sigma^*_x + p^*) \frac{db^*}{dx^*} + U_0 q^* = 0,
$$

(9)

while the strip profile (5) is now determined by the equation

$$
b^*(x^*) = 1 - \frac{\varepsilon}{2 \eta_0} \frac{x'^2 - x^*_0}{2 \eta_0} + \frac{2 \varepsilon}{\pi \eta_0} \int_{x^*_0}^{x^*} p(x') \ln \left| \frac{x^*_0 + x'}{x^*_0 - x'} \right| dx',
$$

(10)

where $\eta_0 = b_0/(R \varepsilon)$ (the scaled ratio of the strip thickness to the radius of the rolls).

\(^\text{1}^\text{Typically about } 10^{-3} \text{ in foil rolling.}\)
In terms of the new variables, the Tresca yield criterion (2) becomes
\[ \sigma^*_x + p^* = 1, \]  
(11)
while Hookes Law (3) is given by
\[ \frac{b^* - b'^*}{b'^*} = -\varepsilon(p + \kappa(\sigma^*_x - \sigma'^*_x)). \]  
(12)
Coulomb’s law (4) now takes the form
\[ q^* = \pm p^*. \]  
(13)
For the remainder of this paper the asterisks will be dropped for notational convenience.

4. Flattened region

In the Influence Function Model, the strip is assumed to be flat in the central flattened region, so that the strip thickness, \( b \), is constant, \( b_1 \), say. We then need to solve the integral equation (10) in the form
\[ b_1 = 1 - \varepsilon \frac{x^2_0 - x^2}{2\eta_0} + \frac{2\varepsilon}{\pi \eta_0} \int_{x_0}^{x_2} p(x') \ln \left| \frac{x_0 + x'}{x - x'} \right| \, dx' \]  
(14)
for the pressure, \( p(x) \), in the flattened region. However, we note that since we assuming the strip thickness is uniform then we could alternatively state that the derivative of the strip thickness should be zero, namely,
\[ \frac{db}{dx} = 0. \]  
(15)
An expression for the derivative of the strip thickness can be found by differentiating Eq. (14),
\[ \frac{db}{dx} = \frac{\varepsilon}{\eta_0} \left( x - \frac{2}{\pi} \int_{x_0}^{x_2} p(x') \frac{x - x'}{x - x'} \, dx' \right) \]  
(16)
so that we then need to satisfy
\[ x = \frac{2}{\pi} \int_{x_0}^{x_2} \frac{p(x')}{x - x'} \, dx' = 0 \]  
(17)
in the flattened region.

By noting that this equation holds only in the central flattened region \( (x_{f1} < x < x_{f2}) \) we take contributions of the integral due to pressure in the regions before and after the flat zone to the right-hand side of the equation. With appropriate scaling, this equation can be written in the form
\[ \frac{1}{\pi} \int_{-1}^{1} \frac{p(z)}{w - z} \, dz = g(w), \]  
(18)
which is known as the Aerofoil Equation and was first discussed by Prandtl [20] who was studying the lift on aerofoils. In this equation we have set \( w = (x - m)/l \) where \( m \) is the mid-point of the flat region, \( m = (x_{f2} + x_{f1})/2 \), and \( l \) is half the length of the flat region, \( m = (x_{f2} - x_{f1})/2 \). In our case
the function $g(w)$ is given by

$$g(w) = \frac{1}{\pi} \left( \frac{\pi}{2} (m + lw) - \int_{-w}^{-1} \frac{p(z)}{w - z} \, dz - \int_{1}^{w} \frac{p(z)}{w - z} \, dz \right). \tag{19}$$

The solution of the Aerofoil Equation, also known as the inverse Hilbert Transform, is [20]

$$p(w) = -\frac{1}{\pi \sqrt{1 - w^2}} \int_{-1}^{1} \frac{g(s) \sqrt{1 - s^2}}{w - s} \, ds + \frac{\Gamma}{\sqrt{1 - w^2}}, \tag{20}$$

where $\Gamma$ is arbitrary constant which represents the flattened region’s contribution to the scaled roll force, that is,

$$\Gamma = \int_{-1}^{1} p(w) \, dw. \tag{21}$$

On substituting Eq. (19) into Eq. (20), the pressure in the flattened region can be found\(^2\)

$$p_f(w) = \frac{2}{\pi} \sum_{w_i < -1} p_i(w) \left( \tan^{-1} \left( \mu_i \sqrt{\frac{1 - w}{1 + w}} \right) - \tan^{-1} \left( \mu_{i+1} \sqrt{\frac{1 - w}{1 + w}} \right) \right) + \frac{2}{\pi} \sum_{w_i \geq 1} p_i(w) \left( \tan^{-1} \left( \nu_{i+1} \sqrt{\frac{1 + w}{1 - w}} \right) - \tan^{-1} \left( \nu_i \sqrt{\frac{1 + w}{1 - w}} \right) \right) + \frac{1}{\pi \sqrt{1 - w^2}} \left\{ \int_{-w}^{w} p(z) \, dz - mw \frac{\pi}{2} - \frac{\pi}{2} \left( w^2 - \frac{1}{2} \right) \right\} + \sum_{w_i < -1} \int_{w_i}^{w_i+1} p_i(w) - p_i(z) \sqrt{z^2 - 1} \, dz + p_i(w) \int_{w_i}^{w_i+1} \frac{z + w}{\sqrt{z^2 - 1}} \, dz - \sum_{w_i \geq 1} \int_{w_i}^{w_i+1} p_i(w) - p_i(z) \sqrt{z^2 - 1} \, dz + p_i(w) \int_{w_i}^{w_i+1} \frac{z + w}{\sqrt{z^2 - 1}} \, dz \right\}, \tag{22}$$

where

$$\mu_i = \sqrt{\frac{w_i + 1}{w_i - 1}} \quad \text{and} \quad \nu_i = \sqrt{\frac{w_i - 1}{w_i + 1}}. \quad \text{Here} \ w_i \text{is the position of the zone boundary} \ x_i \text{in the new scaling. Note in expression (22), the roll force} \ \int_{w}^{w_i} p(z) \, dz \text{has absorbed the} \ \Gamma \text{term in expression (20). This solution requires that we know the pressure profiles,} \ p_i(w), \text{before} \ (w_i < -1) \text{and after} \ (w_i \geq 1) \text{the flattened region. This solution can be written in the form}$$

$$p_f(w) = f(w) + \frac{h(w)}{\sqrt{1 - w^2}}, \quad \text{(23)}$$

which is unbounded at $w = \pm 1$ unless $h(w)$ is zero there. Hence, the difficulty associated with solving the integral equation numerically is due to the presence of these singularities and not

\(^2\)See Langlands [21] for the full derivation.
necessarily the numerical technique used. The infinite pressure values will, in general, appear at the ends of the central flattened region. However, we would like both the pressure and the derivative of the pressure to be finite and continuous. Note we require continuity of the pressure profile to ensure the derivative of the strip profile remain continuous. The continuity of the derivative of the pressure is also required to ensure the shear stress is continuous.

5. Boundary conditions

To maintain continuity of the pressure we require that \( h(w) \) be zero at both ends of the flattened zone, that is,

\[
h(\pm 1) = 0. \tag{24}
\]

Both of these conditions can be satisfied by choosing a particular value of the roll force. However, we will find two different expressions for the roll force derived from the conditions at \( w = 1 \) and \( -1 \). To be consistent, these expressions for the roll force must be the equal. This reduces the two above conditions to (see Appendix A for the derivation)

\[
m \pi \left( \sum_{w_i < -1} \int_{w_i}^{w_i+1} \frac{p_i(z)}{\sqrt{z^2 - 1}} \, dz + \sum_{w_i > 1} \int_{w_i}^{w_i+1} \frac{p_i(z)}{\sqrt{z^2 - 1}} \, dz \right) = 0. \tag{25}
\]

Here \( \sum_{w_i < -1} (\sum_{w_i > 1}) \) denotes a summation over intervals to the left (right) of the flat zone. To find the condition for continuity of the derivative of the pressure, we first evaluate the derivative of the pressure which can be shown to be of the form (23). To remove the singularities at \( w = \pm 1 \) we likewise set the equivalent of \( h(-1) \) and \( h(1) \) to zero. This gives us two more equations to be satisfied (see Appendix A for the derivation).

In addition to these conditions, we also need to satisfy the conditions that the strip thickness must equal to the exit thickness at the exit of the roll gap, the derivative of the strip must be equal to zero at the start of the elastic recovery region, and the strip must yield both at the end of the elastic compression region and at the start of elastic recovery region. This brings the total number of equations and unknown boundary points to seven. This is for the case when there is a flat zone. If there is no flat zone, then the three equations derived for the flattened zone are replaced with the condition that the pressure must be continuous at the neutral point.

6. Pressure approximation

The pressure profiles in each zone, besides the flattened region, are approximated by a quadratic function of the form:

\[
p(x) = p_i + \frac{p_{i+1} - p_i}{x_{i+1} - x_i}(x - x_i) + c(x - x_i)(x - x_{i+1}), \tag{26}
\]

where \( p_i \) and \( p_{i+1} \) are the pressure values at the start, \( x_i \), and at the end \( x_{i+1} \) of the region, respectively. The constant, \( c \), is then chosen to match the load of the numerically generated profile with the area underneath the quadratic function.
This form of approximation has the advantage that there is only one contribution for each zone (4 zones if there is no flat zone or 6 zones if there is a flat zone) in the roll gap when evaluating the influence function, which, with this quadratic representation of the pressure, can be found in closed form. If however, the pressure is found using a full numeric method with many subintervals there would be one contribution for each subinterval. This leads to significantly more computation. However, fitting the pressure by quadratic profiles may be as or more expensive then using the full numerical method. Here the evaluation of the strip profile is performed using a matrix multiplication where the coefficient matrix is constant on a fixed mesh.

6.1. Elastic and plastic regions

In the elastic and plastic regions, the pressure profiles are generated by solving the equilibrium equation numerically, coupled with either the yield condition (plastic region) or Hooke's Law (elastic region), using the current estimate for the roll profile. The constant, $c$, is then chosen to match the load.

7. Finding the minimum of the roll profile

The minimum of the roll profile is found by searching for zeros in the derivative of the roll profile from either the start of the roll gap (if there is no flat region) or the end of the flattened region. If the first minimum occurs before the centre-line of the rolls ($x = 0$) then a flat region is assumed to exist. The position of the last (or only) minimum of the strip profile is used as the approximate location of start of the elastic recovery region.

8. Approximation of strip profile in the elastic recovery region

If the roll indents after the centre-line of the rolls but before the end of the roll gap then the strip profile is assume to be incorrect in this region. To see this we must understand the physics of the problem. If the pressure is decreasing (off-loading) then we expect the roll deflection to likewise decrease, i.e., we do not expect the roll to deform again and pressure to re-increase. The strip profile is then approximated by

$$b(x) = b(x_m) + (b(x_2) - b(x_m)) \left( \frac{x - x_m}{x_2 - x_m} \right)^2,$$

where $x_m$ is the first minimum found after both the centre-line of the rolls and the end of the flat zone (if this is one). This approximation of the strip profile is used to generate the pressure in the elastic recovery region instead of using the actual strip profile.
9. Solution method

The approximate method starts by assuming that the rolls are circular. The position of the minimum of the roll profile, \(x_m\), is known in this case. The following steps are taken in an iteration of the approximate method (see Fig. 1 for the location of zone boundaries, \(x_i\)):

1. Find the end of the elastic compression region, \(x_c\), by integrating the equilibrium equation coupled with Hooke’s Law from the start of the roll gap, \(x_0\), until the yield criterion is satisfied. Evaluate the pressure value, \(p_c\), at \(x_c\).
2. Find the end of the roll gap, \(x_2\), by integrating the equilibrium equation coupled with Hooke’s Law from an estimate of the end of the roll gap, \(x_2\), to the minimum point, \(x_m\), such that the yield criterion is satisfied at the minimum. Evaluate the pressure value, \(p_m\), at the minimum.
3. Integrate the equilibrium equation coupled with the yield criterion and positive friction sign from \(x_c\) to \(x_m\).
4. Integrate the equilibrium equation with the yield criterion and negative friction sign from \(x_m\) until it intersects the curve in the previous step. This gives the position of the neutral point, \(x_{np}\) (if there is no flat region).
5. If a flat region has been shown to exist, solve the set of three equations for the continuity of the pressure profile and its derivative. This gives the position of the neutral point, \(x_{np}\), and the positions of the start, \(x_{f1}\), and end, \(x_{f2}\), of the flat region.
6. Calculate the strip profile using the new estimate of the pressure profile.
7. Under-relax the strip profile \(b^* = b^{(i-1)}(1 - \gamma) + b^{(i)}\gamma\), where \(\gamma\) is the relaxation parameter.
8. Find the minima of the new profile. If there is currently no flat in the roll profile and a minimum exists prior to the centre-line of the rolls then a flat region is assumed to exist in the next iteration.
9. Find the new start of the roll gap by solving \(b(x_0) = b(x_2) + 1 - b_2\) where \(b_2\) is the required strip thickness at the exit of the roll gap.
10. Check the convergence. If the strip profile has not converged, go to step 1.

Note in this method, the flat zone is treated as a single zone and does not distinguish between contained plastic and elastic no-slip zones. The above approach can be modified to take into account strain hardening. In essence, the numerical solution of the pressure profile is modified and as such so are the constants in the quadratic approximation (though in some cases a better approximation may be required). The rest of the steps remain intact.

10. Results

The advantage of the approximate model is that, unlike the full numerical solution, the boundary conditions for the continuity of the pressure and the derivative of the pressure are defined. In the full numerical solution we try to match these values but this causes numerical problems. For instance, in the case of the continuity of the pressure we evaluate the difference in the pressure predictions gained by solution in the central flat and plastic reduction zones—in reality this function will be unbounded but the numerical solution gives inaccurate finite values at
the ends of the central flattened zone. By using the Inverse Hilbert Transform to solve the integral equation we obtain the solution for the pressure in the central flattened region in closed form.

The approximate method was tested on six test cases given in Fleck et al. and compared with the predictions of the commercial package Interactive Rollgap Model described by Domanti et al. [22]. This package solves the equilibrium equations and the corresponding elasticity and plasticity

Fig. 3. Comparison of results for the full numerical solution (×) and the approximate method (—) developed in this paper for the 50% reduction of strip the thickness of (a) 0.14 mm, (b) 0.06 mm, (c) 0.042 mm, (d) 0.03 mm, (e) 0.02 mm and (f) 0.013 mm. Both pressure and strip profiles are shown. The parameter values for each case are given in the text. In cases (a) and (b), the full numerical solution and the results from the approximate method are not distinguishable on the scale used here.
constitutive equations numerically to estimate the strip stresses. A Jortner influence function is used to describe the roll deformation. The test cases were the 50% reduction of aluminium strip by steel rolls. The cases examined are for foil inlet thicknesses of (a) 0.14 mm, (b) 0.06 mm, (c) 0.048 mm, (d) 0.03 mm, (e) 0.02 mm, and (f) 0.013 mm. These results are shown in Figs. 3(a)–(f).

Typical values of the parameters encountered in foil rolling were used: $R = 89$ mm, $\mu = 0.03$, $v_s = 0.3$, $Y_s = 230$ MPa and $E^* = 230$ GPa $= 3E^*_s$. Zero entry and exit tensions were assumed in each case.

We see from these Fig. 3 that there is good agreement between the pressure and the strip profiles predicted by the new approximation method and those profiles predicted by the commercial package. In test cases (a)–(c) and (e) there is little difference between the predicted profiles. In test case (c) the deviation is discernible in flat zone of the strip thickness profile. In the other two cases (d) and (f) the difference between the two predictions is more evident.

This is reflected in the estimates of roll force, contact length and forward slip given in the tables below. It should be pointed out that since the model predictions are based on the estimated strip profile the subsequent zone boundaries are sensitive to the strip profile. Differences in the strip and pressure profiles are due in part to the approximation of the pressure by quadratic functions in evaluating the strip profile.

The predicted values of the roll force, slip and contact length are given in Tables 1–3. The predictions of the commercial package are given as a comparison. The error in these values are small for all test cases. The largest error being in test case (f) for the predictions of roll force, test case (d) for contact length and case (c) for the prediction of forward slip.

### Table 1
Comparison of roll force predictions (kN/mm) for the 50% reduction of strip the thickness of (a) 0.14 mm, (b) 0.06 mm, (c) 0.042 mm, (d) 0.03 mm, (e) 0.02 mm and (f) 0.013 mm. The relative error is also given

<table>
<thead>
<tr>
<th>Case</th>
<th>Numerical method</th>
<th>Approximate method</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0.954</td>
<td>0.951</td>
<td>0.36</td>
</tr>
<tr>
<td>b</td>
<td>1.375</td>
<td>1.375</td>
<td>0.04</td>
</tr>
<tr>
<td>c</td>
<td>2.025</td>
<td>2.022</td>
<td>0.16</td>
</tr>
<tr>
<td>d</td>
<td>4.112</td>
<td>3.987</td>
<td>3.15</td>
</tr>
<tr>
<td>e</td>
<td>6.955</td>
<td>6.928</td>
<td>0.40</td>
</tr>
<tr>
<td>f</td>
<td>10.324</td>
<td>9.915</td>
<td>3.96</td>
</tr>
</tbody>
</table>

### Table 2
Comparison of forward slip predictions (%) for the 50% reduction of strip the thickness of (a) 0.14 mm, (b) 0.06 mm, (c) 0.042 mm, (d) 0.03 mm, (e) 0.02 mm and (f) 0.013 mm. The relative error is also given

<table>
<thead>
<tr>
<th>Case</th>
<th>Numerical method</th>
<th>Approximate method</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1.066</td>
<td>1.067</td>
<td>0.14</td>
</tr>
<tr>
<td>b</td>
<td>1.197</td>
<td>1.206</td>
<td>0.72</td>
</tr>
<tr>
<td>c</td>
<td>1.357</td>
<td>1.387</td>
<td>2.24</td>
</tr>
<tr>
<td>d</td>
<td>1.481</td>
<td>1.498</td>
<td>1.17</td>
</tr>
<tr>
<td>e</td>
<td>1.555</td>
<td>1.562</td>
<td>0.48</td>
</tr>
<tr>
<td>f</td>
<td>1.647</td>
<td>1.641</td>
<td>0.33</td>
</tr>
</tbody>
</table>
The discrepancies in pressure profiles seen in Figs. 3(d) and (f) explain the larger errors seen in the roll force and contact length predictions. The estimates of slip, however, are dependent on the strip thickness at the neutral point (Regime I) or the strip thickness in the flat zone (Regimes II and III). So it is not surprising that test cases (c) and (d) have the worst predictions as the difference in the flat region’s strip thickness can be seen in Figs. 3(c) and (d).

11. Conclusion

We have developed an alternative approach to the solution of the Fleck et al. influence function model based on the use of Inverse Hilbert Transform to solve the governing integral equation in the flattened zone. The advantage is that this explicit form shows the form of the pressure profile in this zone explicitly and indicates how and where numerical problems can arise.

The approximations we have introduced have only resulted in small errors in the test cases investigated and with better approximations than quadratic functions could be sought to improve accuracy. However, in these test cases the friction coefficient and yield stress are assumed constant and strain hardening is ignored. These test cases may not adequately reflect the operating conditions in the rolling mill. In the mill, the greatest sources of modelling error are in fact the yield stress, the friction conditions and to a lesser extent strip temperature variation. The approach taken in the approximate method can be easily adapted to take into account these effects making the model more realistic. However, this will require more information to estimate the parameters of the more sophisticated yield and friction models which may not be available. It also should be noted that in the rolling mill the roll force and torque are measured so the accuracy is instrumentation-dependent and will contain some error.

The authors acknowledge that the Fleck et al. influence model [7] (from which this approximation is based) is not the most sophisticated model available for rolling. Effects of temperature variation, roll speed, and plastic working are not included in the model and whilst the influence model is simple, it still poses numerical problems that currently prohibit its use for on-line control.

Although based on a simple model, the approximation method developed in this paper is an important first step to achieving efficient on-line software. The method can be easily adapted to incorporate strain hardening and temperature effects by making suitable changes to the governing equations.

**Table 3**

Comparison of contact length predictions (mm) for the 50% reduction of strip the thickness of (a) 0.14 mm, (b) 0.06 mm, (c) 0.042 mm, (d) 0.03 mm, (e) 0.02 mm and (f) 0.013 mm. The relative error is also given.

<table>
<thead>
<tr>
<th>Case</th>
<th>Numerical method</th>
<th>Approximate method</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>3.017</td>
<td>3.018</td>
<td>0.03</td>
</tr>
<tr>
<td>b</td>
<td>2.416</td>
<td>2.413</td>
<td>0.10</td>
</tr>
<tr>
<td>c</td>
<td>2.513</td>
<td>2.510</td>
<td>0.10</td>
</tr>
<tr>
<td>d</td>
<td>3.096</td>
<td>3.039</td>
<td>1.85</td>
</tr>
<tr>
<td>e</td>
<td>3.817</td>
<td>3.824</td>
<td>0.19</td>
</tr>
<tr>
<td>f</td>
<td>4.532</td>
<td>4.468</td>
<td>1.43</td>
</tr>
</tbody>
</table>
Appendix A. Boundary conditions

In this appendix we derive the necessary conditions to ensure continuity of pressure at both ends of the flat zone and continuity of the derivative at the start of the flat zone.

To maintain continuity of the pressure we require that \( h(w) \) to be zero at both ends of the flattened zone:
\[
h(\pm 1) = 0, \tag{A.1}\]
that is, at \( w = 1 \) we require that the roll force is set as
\[
\int_{w_1}^{w_2} p(z) \, dz = m \frac{\pi}{2} + l \frac{\pi}{4} - \sum_{w_j < -1} \int_{w_j}^{w_{j+1}} \frac{p_j(1) - p_j(z)}{1 - z} \sqrt{z^2 - 1} \, dz \\
- \sum_{w_i < -1} p_i(1) \int_{w_i}^{w_{i+1}} \frac{z + 1}{\sqrt{z^2 - 1}} \, dz + \sum_{w_j \geq 1} p_j(1) \int_{w_j}^{w_{j+1}} \frac{z + 1}{\sqrt{z^2 - 1}} \, dz \\
+ \sum_{w_i \geq 1} \int_{w_i}^{w_{i+1}} \frac{p_i(1) - p_i(z)}{1 - z} \sqrt{z^2 - 1} \, dz \tag{A.2}
\]
and at \( w = -1 \)
\[
\int_{w_1}^{w_2} p(z) \, dz = -m \frac{\pi}{2} + l \frac{\pi}{4} - \sum_{w_j < -1} \int_{w_j}^{w_{j+1}} \frac{p_j(-1) - p_j(z)}{-1 - z} \sqrt{z^2 - 1} \, dz \\
- \sum_{w_i < -1} p_i(-1) \int_{w_i}^{w_{i+1}} \frac{z - 1}{\sqrt{z^2 - 1}} \, dz + \sum_{w_j \geq 1} p_j(-1) \int_{w_j}^{w_{j+1}} \frac{z - 1}{\sqrt{z^2 - 1}} \, dz \\
+ \sum_{w_i \geq 1} \int_{w_i}^{w_{i+1}} \frac{p_i(-1) - p_i(z)}{-1 - z} \sqrt{z^2 - 1} \, dz, \tag{A.3}
\]
which gives two expressions for the roll force. To be consistent the roll force must be the equal in both equations. This reduces the two above conditions to
\[
m \frac{\pi}{2} - \sum_{w_j < -1} \int_{w_j}^{w_{j+1}} \frac{p_j(z)}{\sqrt{z^2 - 1}} \, dz + \sum_{w_j \geq 1} \int_{w_j}^{w_{j+1}} \frac{p_j(z)}{\sqrt{z^2 - 1}} \, dz = 0. \tag{A.4}
\]
To find the condition for continuity of the derivative of the pressure, we first evaluate the derivative of the pressure, namely,
\[
\frac{dp_j}{dw}(w) = \frac{2}{\pi} \sum_{w_j < -1} \frac{dp_j}{dw}(w) \left( \tan^{-1} \left( \mu_i \sqrt{\frac{1 - w}{1 + w}} \right) - \tan^{-1} \left( \mu_{i+1} \sqrt{\frac{1 - w}{1 + w}} \right) \right) \\
+ \frac{2}{\pi} \sum_{w_i \geq 1} \frac{dp_i}{dw}(w) \left( \tan^{-1} \left( \nu_i \sqrt{\frac{1 + w}{1 - w}} \right) - \tan^{-1} \left( \nu_{i+1} \sqrt{\frac{1 + w}{1 - w}} \right) \right)
\]
To remove the singularities we need the bracketed term, \{\}^*, set to zero at \(w = \pm 1\). Using l’Hôpital’s Rule we obtain

$$\lim_{w \to \pm 1} \frac{w}{1 - w^2} h(w) = -\frac{1}{2} \frac{dh}{dw}(1)$$  \hspace{1cm} (A.6)

and

$$\lim_{w \to \pm 1} \frac{w}{1 - w^2} h(w) = -\frac{1}{2} \frac{dh}{dw}(-1)$$  \hspace{1cm} (A.7)

so the conditions \{\}^* = 0 at \(w = \pm 1\) become

$$\frac{1}{2} \frac{dh}{dw}(1) - \sum_{w_j < -1} p_j(1)(\mu_i - \mu_{i+1}) + \sum_{w_j \geq 1} p_j(1)\left(\frac{1}{v_{i+1}} - \frac{1}{v_i}\right) = 0,$$  \hspace{1cm} (A.8)

$$\frac{1}{2} \frac{dh}{dw}(-1) - \sum_{w_j < -1} p_j(-1)\left(\frac{1}{\mu_i} - \frac{1}{\mu_{i+1}}\right) + \sum_{w_j \geq 1} p_j(-1)(v_{i+1} - v_i) = 0.$$  \hspace{1cm} (A.9)

References


