

CHAPTER 10

INTRODUCTION TO COMPRESSIBLE FLOW

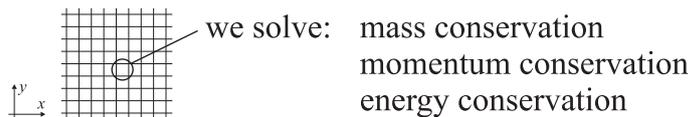
- Collapsing PDEs to ODEs
- The Steady Flow Energy Equation
- The enthalpy of a fluid
- Stagnation enthalpy and stagnation temperature
- Entropy
- Viscous dissipation, entropy and irreversibility
- Transfer from thermal energy to mechanical energy
- Incompressible flows
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10.1 INTRODUCTION

So far we have examined *incompressible* flows, which have the equation of state $\rho = \text{constant}$. In this chapter we will examine *compressible* flows, which have other equations of state. For simplicity, we will examine *perfect* gases, which have the equation of state $\rho = p/RT$ and for which c_p , c_v , and R are constant. We will discover that, if the flow is incompressible, the thermodynamics can be separated from the kinematics (the movement of the flow) and dynamics (the forces in the flow). It is worth reviewing the 1A thermodynamics course, which covers this topic in detail.

10.2 COLLAPSING PDEs TO ODEs

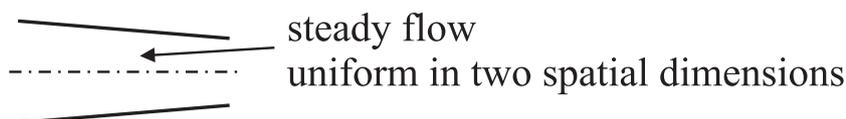
In chapters 1 and 2 we used partial differential equations for mass ($\partial\rho/\partial t = \nabla \cdot \rho\mathbf{v}$) and momentum (Euler's equation) in order to evaluate the pressure and velocity fields in three spatial dimension and one time dimension. In compressible flows, the energy field must also be evaluated in these dimensions. This is the basis of full CFD (Computational Fluid Dynamics) solvers.



Partial differential equations are usually impossible to solve by hand. In the rest of this course, two special cases will be considered, both of which reduce the partial differential equations (PDEs) to ordinary differential equations (ODEs).



One special case is *unsteady flow in zero spatial dimensions*. All quantities are assumed to be uniform¹ within a closed system. This *quasi-equilibrium* assumption means that all the spatial derivatives are zero and therefore the time derivative $\partial/\partial t$ collapses to d/dt . Alternatively, the time derivative can be replaced with a derivative with respect to some other variable that measures progress. A further simplification is simply to examine the start and end states of the closed system.

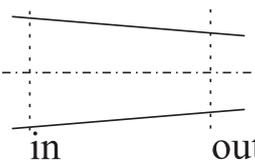


The other special case is *steady flow in one spatial dimension*. All quantities are assumed uniform in two spatial dimensions and constant in time. This means that only, say, the x -derivative is non-zero and therefore that $\partial/\partial x$ collapses to d/dx . The x coordinate can then be taken to be the distance along a streamline.

¹'uniform' means 'having the same value at every point in space'

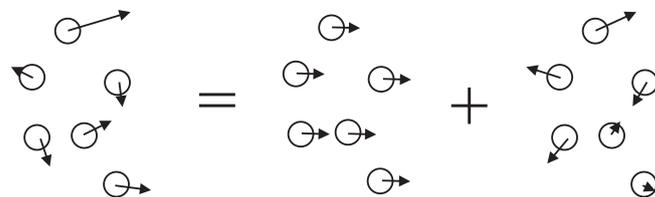
10.3 THE STEADY FLOW ENERGY EQUATION (SFEE)

In a further simplification, we examine the differences between entry and exit states of a fluid passing through a finite-size control volume. This leads to the steady flow energy equation, which relates changes in the total energy of a fluid across a control volume to the heat input to and work done (both shaft work and displacement work) by the fluid within the control volume:



$$\dot{m} \left(u + \frac{p}{\rho} + \frac{v^2}{2} + gz \right)_{out} - \dot{m} \left(u + \frac{p}{\rho} + \frac{v^2}{2} + gz \right)_{in} = Q - W_x$$

We will consider each of these terms in turn. Within a perfect gas, the molecules have an average velocity, \bar{v} , which is called the *bulk* velocity. Per unit mass, the kinetic energy of this *ordered* motion is $|\bar{v}|^2/2$, which we shall write as $v^2/2$ by defining $v = |\bar{v}|$. It is a mechanical energy.



total kinetic energy of
molecular motion =

Much of the molecules' energy, however, is contained in *disordered* motion. For a perfect gas, the kinetic energy per unit mass of this disordered motion is the *internal energy*², u . The increase in u is proportional to the increase in T according to $du = c_v dT$. It is a thermal energy.

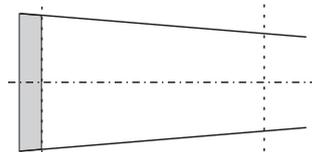
The gravitational potential energy per unit mass is gz . It is a mechanical energy.

²The internal energy also includes any potential energy due to inter-molecular forces but these are negligible in ideal gases and perfect gases

Conceptually, we have to be careful with the term p/ρ . This term can be written $p\nu$, where, in this paragraph, ν is the specific volume, $1/\rho$. In IA thermodynamics you applied a control volume analysis to the first law for a system (§11.2 *The Conservation of Energy for a Control Volume*). You saw that changes in $p\nu$ are caused by displacement work $p\delta\nu$ and shaft work which, if reversible, is $\nu\delta p$:

$$\delta\left(\frac{p}{\rho}\right) = \delta(p\nu) =$$

The shaft work, W_x , appears explicitly in the SFEE and it is easy to see how it affects the energy change across the control volume. The displacement work, however, is more subtle.



For simplicity, let us set $W_x = 0$ and $Q = 0$. The fluid in the control volume upstream does displacement work on the fluid going into the control volume that we are looking at. Similarly, the fluid in the control volume that we are looking at does displacement work on the fluid going into the control volume downstream. In the absence of shaft work, the total work done by the fluid (per unit mass) is equal to the downstream p/ρ minus the upstream p/ρ .

With $W_x = 0$ and $Q = 0$, a reduction in p/ρ from one side of the control volume to the other will be matched by an increase in $u + v^2/2 + gz$. Therefore, in loose terms, p/ρ , which is a thermodynamic property of the fluid, can be thought of as another mechanical energy held by the fluid.

10.4 THE ENTHALPY OF A FLUID

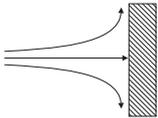
The total energy per unit mass of the fluid is $u + v^2/2 + gz$. If the p/ρ term is included we obtain the quantity that we follow in the steady flow energy equation:

$$u + \frac{p}{\rho} + \frac{v^2}{2} + gz$$

The first two terms are thermodynamic properties of the *fluid*, while the second two are kinematic properties of the *flow* (i.e. they relate to the position and velocity of the fluid). For flow processes it is useful to add the first two terms together to make the *enthalpy*, h . It is not an energy. It is simply the shorthand for $u + p/\rho$.

10.5 STAGNATION ENTHALPY AND STAGNATION TEMPERATURE

If no heat is transferred to the fluid and no work is done on it, $u + p/\rho + v^2/2 + gz$ remains constant. For instance, if a fluid slows to zero velocity without changing height, all its kinetic energy of ordered motion $v^2/2$ is transferred into enthalpy. This allows us to define the stagnation enthalpy and stagnation temperature:



In some books, the gravitational potential energy gz is included in the stagnation enthalpy and stagnation temperature. The stagnation temperature then includes an extra gz/c_p term and is no longer simply the temperature that is measured at a stagnation point. In this course, the stagnation temperature is defined as $h_0 = h + v^2/2$ and $h_0 + gz$ is constant if no heat or work is done on the fluid.

The Tds equations give:

$$Tds = dh - \frac{1}{\rho} dp$$

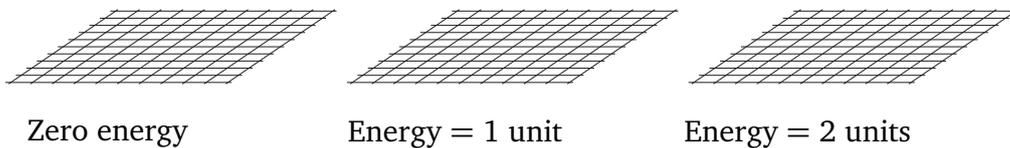
Re-arranging, we discover that the pressure recovery in a flow process is related to the enthalpy change and entropy change by:

In other words, the pressure recovery is maximized when there are no irreversibilities in the flow - i.e. when the entropy change due to irreversibilities is zero.

10.6 ENTROPY

In loose terms, entropy, which is given the symbol s , measures the degree of disorder of a system. More exactly, $s = k \ln \Omega$, where k is the Boltzmann constant and Ω is the number of microstates of the system. In a fluid, Ω is the number of ways in which all the molecules could exist, given macroscopic constraints such as its total energy and volume. In other words, entropy measures our uncertainty about the speeds and locations of the individual molecules in a fluid.

For illustration, let us put fluids to one side and consider instead an imaginary system of 100 particles that have discrete energy levels.

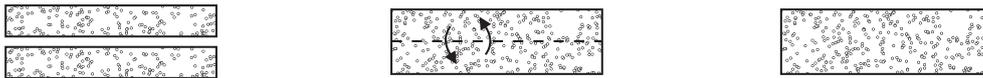


At zero energy, this imaginary system has one possible microstate: $\Omega = 1$. When just enough internal energy (thermal energy) is added to push one particle up one energy level, there are 100 possible microstates: $\Omega = 100$. When the same amount of internal energy is added again, there are 100×100 possible states (9900 states with two molecules up one level and 100 states with one molecule up two levels): $\Omega = 10,000$. The entropy increases as the internal energy increases because, with more energy, the particles in the imaginary system can occupy more microstates.

In a fluid, the treatment is more complicated. It is covered in the 4th year course: *4A9 Molecular Thermodynamics*.

10.7 VISCOUS DISSIPATION, ENTROPY AND IRREVERSIBILITY

When there is no exchange of heat with the surroundings, the entropy within a fluid can never decrease, but it can increase. Let us imagine a system containing two layers of fluid, both of mass m and with bulk velocities v_1 and v_2 . By molecular collisions they exchange momentum and eventually come to the same bulk velocity $(v_1 + v_2)/2$. This process, which we met in chapter 3, is known as viscous dissipation³.



$$= \frac{1}{2}(2m) \left(\frac{v_1 + v_2}{2} \right)^2 - \frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2 = -m \left(\frac{v_1 - v_2}{2} \right)^2$$

Because of this process, the kinetic energy of *ordered* motion has decreased. This might seem odd because we know that there cannot be any loss of kinetic energy at the molecular level (if there were, gases would cool down by themselves). In fact, after the mixing, the molecules are whizzing around a little faster than they were before and therefore the kinetic energy of *disordered* motion, u , has increased by the same amount. This process increases the temperature of the fluid.

In a *flow* process, the kinetic energy of ordered motion is transferred to the *enthalpy*. However, the *stagnation* enthalpy and the *stagnation* temperature, which both contain the $v^2/2$ term, remain constant.

The net result is a decrease in mechanical energy, an equal increase in thermal energy and an increase in the degree of disorder of the system. The increase in disorder is measured by the entropy change, which is equal to the shift in energy divided by the temperature, T . This is an *irreversible* process; the increase in disorder cannot be reversed without the surroundings taking heat from or doing work on the fluid.

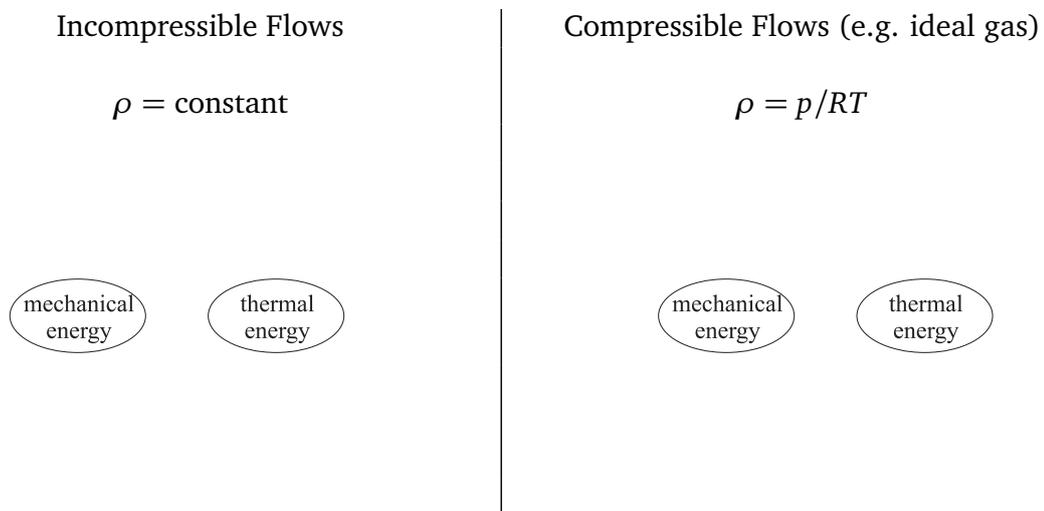
This gives us another useful way to picture entropy. For a flow with a given total energy, the entropy measures the distribution between thermal energy and mechanical energy. The lower the entropy, the greater the proportion of mechanical energy and the greater is the potential to obtain useful work from the flow.

³You met this in section 6.4 of the 1A thermodynamics course, in which it was shown that dissipative processes always entail a lost opportunity for doing mechanical work.

10.8 TRANSFER FROM THERMAL TO MECHANICAL ENERGY

In section 10.7 we saw that energy can shift from mechanical to thermal through viscous dissipation but that this process cannot be reversed. So is there any mechanism that can shift energy from thermal to mechanical?

Consider heating up a point in a stationary fluid, perhaps with a laser. At that point, the thermal energy of the fluid increases, the density drops, the fluid expands and it does work on the surrounding fluid. Thermal energy has been transferred to mechanical energy but *this only happens because the density changes*. Here we see the vital role of the equation of state:



In an incompressible flow, the thermal field cannot affect the mechanical field. This makes incompressible flows *much* easier to analyse than compressible flows. We return to this point in section 10.10.

On the other hand, much of the Thermodynamics course is devoted to clever and efficient ways to transfer thermal energy to mechanical energy with compressible working fluids. Indeed, this was the motivation for the origin of Thermodynamics in the 18th century.

10.9 INCOMPRESSIBLE FLOWS

If the density is assumed to be constant, there is no mechanism by which the thermal energy can shift to mechanical energy. In other words, the thermal field, which is measured by temperature, cannot affect the mechanical fields such as velocity and pressure. Energy can still shift from mechanical to thermal, via irreversible processes such as viscous dissipation, but this is determined entirely by the mechanical field⁴.

We must, however, determine when it is valid to assume that a flow is incompressible. Let us imagine a perfect gas (with temperature T and velocity v) as its velocity changes. Ignoring the gravitational term gz , its stagnation enthalpy is $h + v^2/2$. If no heat or work is exchanged with the surroundings then incremental changes in T and v are related by:

$$h + v^2/2 = \text{const} \Rightarrow dh + v dv = 0 \Rightarrow$$

If the fluid's velocity changes without any viscous dissipation, for example by moving through a pressure gradient, the entropy remains constant. In such an *isentropic* process, $T\rho^{1-\gamma}$ is a constant:

$$\begin{aligned} T\rho^{1-\gamma} &= \text{const} \\ \Rightarrow \frac{dT}{T} + (1-\gamma)\frac{d\rho}{\rho} &= 0 \end{aligned}$$

But we know that $c_p dT + v dv = 0$ so substituting for dT and using the relation $c_p(\gamma - 1) = \gamma R$ gives:

$$\frac{d\rho}{\rho} = - \left(\frac{v^2}{\gamma RT} \right) \frac{dv}{v} =$$

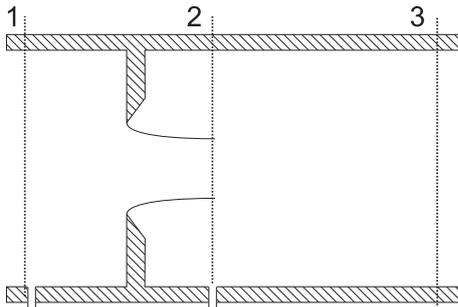
This expression shows us that, in a perfect gas, the density change is very small in steady flows at low Mach number. The same is true for non-perfect gases and for liquids, although it is not formally shown here. In liquids, the speed of sound is so large that almost all liquid flows can be assumed to be incompressible. In gases we need to be more careful:

| | Speed (mph) | Speed (m/s) | M |
|-----------|-------------|-------------|---|
| Bike | | | |
| Car | | | |
| Train | | | |
| Aeroplane | | | |

⁴unless we allow viscosity to be a function of temperature, but this is a small detail

10.10 STAGNATION PRESSURE

In a compressible flow, the stagnation temperature (section 10.5) is a measure of the *total* energy of the fluid⁵ (i.e. thermal and mechanical). The stagnation pressure, is a measure of only the *mechanical* energy of the fluid. If no heat is transferred or work done on a fluid, the stagnation temperature remains uniform throughout. On the other hand, the stagnation pressure reduces if there are irreversibilities in the flow because mechanical energy is lost to thermal energy.



In an incompressible flow, the thermal field cannot affect the mechanical field and we only need to consider the stagnation pressure. The Steady Flow Energy Equation then decouples into a mechanical energy equation and a thermal energy equation:

$$\dot{m}\Delta\left(u + \frac{p}{\rho} + \frac{1}{2}v^2 + gz\right) = Q - W_x$$

$$\dot{m}\Delta\left(\frac{p}{\rho} + \frac{1}{2}v^2 + gz\right) + \text{irreversibilities} = -W_x$$

$$\dot{m}\Delta u - \text{irreversibilities} = Q$$

We have already met the mechanical energy equation in the pipe network analysis of chapter 6. You also met it in section 13.2 of the 1A thermodynamics course. If the flow is incompressible and irreversible and $W_x = 0$, the SFEE applied along a streamline reduces to Bernoulli's equation:

⁵Remember that we did not include the gravitational potential energy gz in this definition because we want 'stagnation temperature' to mean 'the temperature at a stagnation point'.