Application of receptivity and sensitivity analysis to thermoacoustic instability

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1 Introduction

1.1 Receptivity and sensitivity analysis

Receptivity and sensitivity analysis is a branch of linear stability analysis. In stability analysis, one typically calculates a linear system’s eigenmodes. These encapsulate the frequency, growth rate, and mode shape of each natural mode of the system. Receptivity analysis then quantifies the receptivity of each mode to external (open loop) forcing. Sensitivity analysis quantifies the sensitivity of each mode either to internal feedback, which is known as the structural sensitivity, or to changes in the base state, which is known as the base state sensitivity. Sensitivity analysis can be performed by finite difference - e.g. by computing the system’s eigenvalues at two slightly different base states and then calculating the gradient with respect to the change between the two states - but this is computationally expensive and prone to numerical error. A more efficient and more accurate method is to use adjoint equations, which is the subject of this lecture.

The use of adjoint equations in flow instability dates back to the early 1990s (Hill, 1992a,b; Chomaz, 1993; Hill, 1995a) but did not become widespread until the late 2000s (Giannetti and Luchini, 2007; Marquet et al., 2008). The concepts are explained in three review articles (Chomaz, 2005; Sipp et al., 2010; Luchini and Bottaro, 2014), which are essential reading. Pedagogical examples with Matlab tutorials designed for graduate students who are new to the field can be found in Schmid and Brandt (2014).

1.2 Thermoacoustic instability

When a fluctuating source of heat interacts with acoustic waves, for example inside combustion chambers of aeroplane and rocket engines, thermoacoustic oscillations can occur. In the cyclic process created by the acoustic waves, mechanical energy is fed into oscillations over one cycle if higher heat release occurs at moments of higher pressure and lower heat release occurs at moments of lower pressure. This is because the extra heat release at the moments of high pressure causes more work to be extracted during the decompression phase than was required during the preceding compression phase. The amplitude of the thermo-acoustic oscillations therefore grows with time if the heat release fluctuations are within one quarter cycle of the pressure fluctuations. For a linear stability analysis, one typically examines small acoustic perturbations around a steady flow. If these grow in time then this steady flow is thermoacoustically unstable.

Thermoacoustic oscillations were first documented in the 1880s (Rayleigh, 1880, 1878) but at that stage were little more than a curiosity discovered during the glass-blowing process. They became a serious research subject from the late 1930s, particularly during the US Apollo program in the 1960s, during which thermoacoustic oscillations were one of the most important challenges facing the program. Research during this period is extensively reviewed in the early chapters of Culick (2006). Recently, as NOx emissions targets for civil aircraft and power generation have become stricter, manufacturers have attempted to lower the fuel to air ratio in the combustion chambers of gas turbines. This makes flames more receptive to acoustic perturbations and thereby increases their propensity for thermoacoustic oscillations (Lieuwen, 2012). Despite over 60 years of research, thermoacoustic oscillations still present one of the biggest problems facing rocket
and aircraft engine manufacturers. Our intention is that, by applying receptivity and sensitivity analysis in this field, new insights into control of thermacoustic oscillations can be achieved.
2 Direct and adjoint governing equations

2.1 Definition of the direct governing equations

In a linear stability analysis, the direct equations are formed by considering the behaviour of small perturbations around a base state. These perturbations are expressed as a state vector \( \mathbf{q} \) whose time evolution is governed by the direct equation:

\[
A \frac{\partial \mathbf{q}}{\partial t} - L \mathbf{q} = \hat{s} \exp(\sigma_s t),
\]

(1)

where the term on the right hand side is a forcing signal. In this forcing signal, \( \hat{s} \) is the spatial distribution and \( \sigma_s \) is its growth rate / frequency. For a thermoacoustic system, the state vector, \( \mathbf{q} \), is equal to \( (F, u, p)^T \), where \( F \) contains the flame variables, such as the mixture fraction for diffusion flames, \( u \) is the acoustic velocity, and \( p \) is the acoustic pressure.

2.2 Definition of the adjoint governing equations

The adjoint state vector, \( \mathbf{q}^+ \), evolves according to the adjoint equation:

\[
A^+ \frac{\partial^+ \mathbf{q}^+}{\partial t} - L^+ \mathbf{q}^+ = 0,
\]

(2)

If (1) and (2) represent the equations for continuous distributions \( \mathbf{q} \) and \( \mathbf{q}^+ \) then A, L, A\(^+\), and L\(^+\) are operators. In this case, the adjoint operators and equations are analytically derived and then numerically discretized (CA, discretization of Continuous Adjoints). If (1) and (2) represent the equations for numerically discretized systems then A, L, A\(^+\), and L\(^+\) are matrices (in bold from now on) and \( \mathbf{q} = (\mathbf{G}, \eta, \alpha)^T \), where \( \mathbf{G}, \eta, \alpha \) represent the discretization of the heat source, the acoustic velocity and the acoustic pressure, respectively. In this case, the adjoint matrices and functions are directly derived from the numerically discretized direct system (DA, Discrete Adjoints).

When we follow the CA approach, the adjoint systems are defined through a bilinear form\(^1\) \([\cdot, \cdot]\), such that:

\[
\left[ \mathbf{q}^+, \left( A \frac{\partial}{\partial t} - L \right) \mathbf{q} \right] - \left[ \left( A^+ \frac{\partial^+}{\partial t} - L^+ \right) \mathbf{q}^+, \mathbf{q} \right] = \text{constant},
\]

(3)

which, in this paper, defines an inner product\(^2\). For brevity, in this lecture we define the following bracket operators to represent inner products:

\[
\langle \mathbf{a}, \mathbf{b} \rangle = \frac{1}{V} \int_V \mathbf{a}^* \cdot \mathbf{b} \, dV,
\]

(4)

\[
\langle\langle \mathbf{a}, \mathbf{b} \rangle \rangle = \frac{1}{\partial V} \int_{\partial V} \mathbf{a}^* \cdot \mathbf{b} \, d\partial V,
\]

(5)

\[
[\mathbf{a}, \mathbf{b}] = \frac{1}{T} \int_0^T \int_V \mathbf{a}^* \cdot \mathbf{b} \, dV \, dt,
\]

(6)

\(^1\)The calculation of the adjoint function depends on the choice of the bilinear form.

\(^2\)A discussion of possible definitions of the adjoint operator is reported in the supplementary material of Luchini and Bottaro (2014).
where \( a \) and \( b \) are arbitrary functions in the function space in which the problem is defined; \( V \) is the space domain and \( \partial V \) is its boundary; \( t \) is the time; and \( \ast \) is the complex conjugate. Therefore, in this paper, the adjoint operator is defined through the following relation:

\[
\int_T \int_V q^+ \cdot \left( A \frac{\partial}{\partial t} - L \right) q \, dV \, dt - \int_T \int_V \left( A^+ \frac{\partial^+}{\partial t} - L^+ \right)^* q^+ \cdot q \, dV \, dt = \text{constant.} \tag{7}
\]

2.3 Derivation of the adjoint equations from the direct equations

To find the adjoint operator with the CA approach we perform integration by parts of (7). The above relation is an elaboration of the generalized Green’s identity (Dennery and Krzywicky, 1996; Magri and Juniper, 2013c). The adjoint boundary / initial conditions, which arise from integration by parts of (7), are defined such that the constant on the RHS is zero. By integration by parts, we find the important result that \(-A \partial/\partial t = A^+ \partial^+ / \partial t\). Setting \( A^+ = A^* \), then \(-\partial/\partial t = \partial^+ / \partial t\). In other words, the adjoint operator must evolve backwards in time for a problem to be well-posed.

When we follow the DA approach, the adjoint matrix, \( L^+_{ij} \), is defined through the Euclidean product (in Einstein’s notation)

\[
q^+_i L_{ij} q_j - q_i L^+_{ij} q^+_j = 0. \tag{8}
\]

The above terms are scalars, so the transposition does not change the equation. Therefore, we take the transpose of the second term and equate it to the first term:

\[
q^+_i L_{ij} q_j = (q_i L^+_i q^+_j)^T = q^+_i L^+_i q^+_j,
\]

\[
\Rightarrow L^+_i = L^*_{ji}. \tag{10}
\]

This shows that the adjoint matrix is the conjugate transpose of the direct matrix. From now on, when we use the DA method, we denote the direct state vector as \( \chi \) and the corresponding adjoint vector as \( \xi \).

A comparison between the numerical truncation errors between the CA and DA methods is illustrated in Magri and Juniper (2013c). Although the two formulations should converge in principle, it has been shown that convergence is not guaranteed \emph{a priori} (Vogel and Wade, 1995; Bewley, 2001; Pierce and Giles, 2004). For the thermo-acoustic system considered in this paper, the DA method is more accurate and easier to implement. However, we show the results obtained via the CA method in order to describe how the method works.

2.4 The bi-orthogonality condition

In stability/receptivity analysis, we perform a Laplace transform and consider the eigenvalue problems of (1) and (2):

\[
\sigma A \tilde{q} - L \tilde{q} = 0, \tag{11}
\]

\[
\sigma A^+ \tilde{q}^+ - L^+ \tilde{q}^+ = 0. \tag{12}
\]
where \( \hat{q} \) and \( \hat{q}^+ \) are the eigenfunctions, and \( \sigma \) and \( \sigma^+ \) are the eigenvalues. A very important property of the adjoint and direct eigenpairs \( \{\sigma_i, \hat{q}_i\} \) and \( \{\sigma_j^+, \hat{q}_j^+\} \) is the bi-orthogonality condition:

\[
(\sigma_i - \sigma_j^+) \langle \hat{q}_j^+, A\hat{q}_i \rangle = 0, 
\]

which states that the inner product \( \langle \hat{q}_j^+, A\hat{q}_i \rangle \) is zero for every pair of eigenfunctions except when \( i = j \), as long as \( \sigma_j^+ = \sigma_j^* \); in accordance with Salwen and Grosch (1981). This means that the adjoint operator’s spectrum is the complex conjugate of the direct operator’s spectrum. This information serves as good check when validating adjoint algorithms.

### 2.5 Receptivity

Here, we show that the adjoint eigenfunction quantifies the system’s receptivity to open-loop forcing. The receptivity of boundary layers has been calculated from the Orr-Sommerfeld equation by Salwen and Grosch (1981) and Hill (1995b). Another elegant formulation of the receptivity problem, based on the inverse Laplace transform and residues theorem, is described by Giannetti and Luchini (2007, pp. 172–174). A more general approach to the receptivity problem via adjoint equations can be found, among others, in Marino and Luchini (2009, p. 42), Meliga et al. (2009, p. 605), Sipp et al. (2010, p. 10), and Luchini and Bottaro (2014). These studies all concern flow stability. In this lecture, we extend these methods to be able to consider thermoacoustic instability using the formulation by Chandler (2010, pp. 63–68), which is sufficiently general for our purposes.

Let \( q \) be a time-dependent state vector defined in a suitable function space and \( L \) be a linear operator that also encapsulates the boundary conditions. We consider the continuous inhomogeneous linear problem (1), with harmonic forcing at complex frequency, \( \sigma_s \), and initial condition \( q(t = 0) = q_0 \). The general solution of this problem is (in the CA approach):

\[
q = \hat{q}_s \exp(\sigma_s t) + q_d + q_{cs},
\]

where \( \hat{q}_s \) is the spatially varying part of the particular solution, \( q_d = \sum_{j=1}^{N} \beta_j \hat{q}_j \exp(\sigma_j t) \) is the discrete-eigenmodes solution, and \( q_{cs} \) is the continuous spectrum solution. Oden (1979) and Kato (1980) contain rigorous mathematical treatises of spectral decomposition of linear operators. Note that an open loop forcing term, such as \( \hat{s} \exp(\sigma_s t) \), does not change the spectrum of the operator. Assuming that the discrete eigenmodes and continuous spectrum form a complete basis, then the particular and homogenous solutions can be projected onto these spaces. Invoking the adjoint eigenfunction and taking advantage of the bi-orthogonality condition (13), we rearrange (14) as:

\[
q = \sum_{j=1}^{N} \left< \hat{q}_j^+, q_0 \exp(\sigma_j t) + \frac{\hat{s} \exp(\sigma_s t) - \exp(\sigma_j t)}{\sigma_s - \sigma_j} \right> \hat{q}_j^+ + \text{proj}[\hat{q}_s, q_{cs}] \exp(\sigma_s t) + q_{cs},
\]

where \( \text{proj}[\hat{q}_s, q_{cs}] \) is the projection of the forcing term onto the continuous spectrum. The solution (15) is valid for a continuous operator (e.g. Orr-Sommerfeld) in an unbounded or semi-unbounded domain. In these notes we wish to consider reduced-order thermoacoustic systems in which acoustic and combustion domains are bounded. In this case, there is no continuous spectrum, therefore \( \hat{q}_{cs} = 0 \).
The first term of (15) provides a physical interpretation of the adjoint eigenfunction. The response of the \( j^{th} \) component of \( q \) in the long-time limit increases (i) as the forcing frequency, \( \sigma_s \), approaches the \( j^{th} \) eigenvalue, \( \sigma_j \), and (ii) as the spatial structure of the forcing, \( \hat{s} \), approaches the spatial structure of the adjoint eigenfunction, \( \hat{q}_j^+ \). For constant amplitude forcing (\( \text{Re}(\sigma_s) = 0 \)) of a system with one unstable eigenfunction (\( \text{Re}(\sigma_1) > 0 \)) the linear response (15), in the limit \( t \to \infty \), reduces to
\[
q = \left\langle \hat{q}_1^+, q_0 \right\rangle \frac{\hat{q}_1}{\left\langle \hat{q}_1^+, A\hat{q}_1 \right\rangle} \exp(\sigma_1 t).
\]
(16)
This shows that the linear response has the frequency/growth rate, \( \sigma_1 \), and the spatial structure, \( \hat{q}_1 \), of the most unstable direct eigenfunction. Furthermore, the magnitude of this response is determined by the extent to which the spatial structure of the initial conditions, \( q_0 \), and the spatial structure of the forcing, \( \hat{s} \), project onto the spatial structure of the adjoint eigenfunction, \( \hat{q}_1^+ \). In other words, the flow behaves as an oscillator with an intrinsic frequency, growth rate, and shape (Huerre and Monkewitz, 1990) and the corresponding adjoint shape quantifies the sensitivity of this oscillation to changes in the spatial structure of the forcing or initial condition. For constant amplitude forcing acting on a stable system (i.e. with no unstable eigenfunctions), the linear response in the limit \( t \to \infty \) reduces to
\[
q = \sum_{j=1}^{N} \left\langle \hat{q}_j^+, \hat{s} \right\rangle \frac{\hat{q}_j}{\left\langle \hat{q}_j^+, A\hat{q}_j \right\rangle} \exp(\sigma_j t).
\]
(17)
This shows that the linear response is at the forcing frequency, \( \sigma_s \), and that the spatial structure contains contributions from all eigenfunctions, \( \hat{q}_j \). Furthermore, the amplitude of each eigenfunction’s contribution increases (i) as \( \sigma_s \) approaches one eigenvalue, \( \sigma_j \) and (ii) as the spatial structure of the forcing, \( \hat{s} \), approaches the spatial structure of that adjoint eigenfunction, \( \hat{q}_j^+ \). This shows that the sensitivity of the response of each mode to changes in the spatial structure of the forcing is quantified by each (corresponding) adjoint eigenfunction, \( \hat{q}_j^+ \). This is seen most clearly by considering the special case in which the forcing term has \( \sigma_s \to \sigma_j \). By applying l’Hôpital’s rule to (15) for \( t \to \infty \) the solution is
\[
q = \left\langle \hat{q}_j^+, \hat{s} \right\rangle \frac{\hat{q}_j}{\left\langle \hat{q}_j^+, A\hat{q}_j \right\rangle} t \exp(\sigma_j t) .
\]
(18)
The sensitivity of the amplitude of the response to changes in the spatial distribution of the forcing term is the adjoint eigenfunction, \( \hat{q}_j^+ \). This is how we define receptivity.

If we consider the discretized system in the inhomogeneous form (DA approach), seeking solutions of the form \( \hat{\chi} \exp(\sigma_s t) \) then we obtain:
\[
(A\sigma_s - L)\hat{\chi} = \hat{g} + A\hat{\chi}(0),
\]
(19)
where \( \hat{g} \) is the discretized source term \( \hat{s} \). In the discretized system, the spectrum consists of a finite set of points. Assuming that the eigenvectors form a complete set, the solution is decomposed as follows:
\[
\chi = \sum_{j=1}^{N} \alpha_j \hat{\chi}_j.
\]
(20)
Substituting eq. (20) back into the discretized version of eq. (11) and premultiplying by the conjugate adjoint eigenvector, \( \hat{\xi}_j^* \), (which is the conjugate left eigenvector), we obtain

\[
\sum_{j=1}^{N} \hat{\xi}_j^*(-L + A \sigma_j) \alpha_j = \hat{\xi}_j^* (\hat{g} + A \hat{\chi}(0)).
\]

Finally, recalling the bi-orthogonality condition for generalized eigenvalues problems (13), we obtain

\[
\hat{\chi} = \frac{1}{\xi^*} \sum_{j=1}^{N} \frac{1}{\sigma_s - \sigma_j} \hat{\xi}_j^* (\hat{g} + A \hat{\chi}(0)) \hat{\chi}_j
\]

This result is analogous to (15) for discretized systems.

An alternative approach is via constrained optimization, which is particularly suitable to thermoacoustics. In this approach, we define a Lagrangian functional as

\[
L(q, q_0, q^+, q_0^+) = J(q_0, q_\partial V, q) - \left[ q^+, A \partial \partial t q - Lq \right] - \langle q_0^+, q(0) - q_0 \rangle + \ldots \nonumber\]

\[
\ldots - \left\langle q_0^{\partial V}, q(\partial V) - q_0^{\partial V} \right\rangle,
\]

where \( J \) is the cost functional to optimize, which is the eigenvalue \( \sigma \) in this lecture, and \( q_\partial V \) is the boundary condition. The first variation of \( L \), along the generic direction \( \delta \hat{q} \) is defined through the Gâteaux derivative, as

\[
\frac{\delta L}{\delta \hat{q}} = \lim_{\epsilon \to 0} \frac{L(q + \epsilon \delta \hat{q}) - L(q)}{\epsilon}.
\]

By imposing the first variations of \( L \) with respect to the state vector, \( q \), to be zero, we define the adjoint equations (12), whose eigenfunctions can be regarded as Lagrange multipliers from a constrained optimization perspective (Gunzburger, 1997). In a thermoacoustic system, \( u^+ \) is the Lagrange multiplier of the acoustic momentum equation, revealing the locations where the thermo-acoustic system is most receptive to forcing (e.g. acoustic forcing); \( p^+ \) is the Lagrange multiplier of the acoustic energy equation, revealing the locations where the system is most receptive to heat injection; \( F^+ \) is the Lagrange multiplier of the flame equation. If the flame is a fast-chemistry diffusion flame, then \( F^+ \) reveals in which regions the flame is most receptive to species injection (Magri and Juniper, 2014). The adjoint boundary conditions can be interpreted similarly.

### 2.6 Sensitivity

In a sensitivity analysis, one calculates how much an eigenvalue changes when the operator, \( L \), changes slightly. With the CA approach (Continuous Adjoint) we study the continuous system - i.e. before numerical discretization. The direct operator, \( L \), is perturbed to \( L + \epsilon \delta L \). Consequently, the eigenvalues are perturbed to \( \sigma_j + \epsilon \delta \sigma_j \), the direct eigenfunctions to \( \hat{\alpha}_j + \epsilon \delta \hat{\alpha}_j \), and the adjoint eigenfunctions to \( \hat{\chi}_j^* + \epsilon \delta \hat{\chi}_j^* \). We substitute these into the continuous eigenproblem (11) and examine the terms of order \( \epsilon \):

\[
(\sigma_j A - L)\epsilon \delta \hat{\alpha}_j + (\epsilon \delta \sigma_j A - \epsilon \delta L)\hat{\chi}_j = 0
\]
Then we pre-multiply by the corresponding adjoint eigenvector:

\[
\langle \hat{q}^+_j, (\sigma_iA - L)\epsilon\hat{q}_j \rangle + \langle \hat{q}^+_j, (\epsilon\delta\sigma_jA - \epsilon\delta L)\hat{q}_j \rangle = 0
\] (26)

The first term on the left hand side is zero because taking the inner products of \(A\delta\hat{q}_j\) and \(L\delta\hat{q}_j\) with \(\hat{q}^+_j\) extracts only the components that are parallel to \(\hat{q}_j\) (see eqn. (13)), for which \((\sigma_jA - L)\hat{q}_j = 0\). This means that the eigenvalue drift, at order \(\epsilon\), is

\[
\delta\sigma_j = \frac{\langle \hat{q}^+_j, \delta L\hat{q}_j \rangle}{\langle \hat{q}^+_j, A\hat{q}_j \rangle}.
\] (27)

Note that the denominator is always different from zero because the dimension of the adjoint space is equal to the original space’s dimension, under not restrictive conditions Maddox (1988).

With the DA approach (Discrete Adjoint) we study the discretized system, represented by the matrices \(A, L\). From (10) we can infer that the adjoint eigenvector is the conjugate left eigenvector of the system, i.e. \(\hat{\xi}^* \cdot (\sigma_jA - L) = 0\). The bi-orthogonality property ensues directly from definition of right and left eigenvectors

\[
\hat{\xi}^* \cdot A \cdot \hat{\chi}_i = \delta_{ij}.
\] (28)

Now, let us consider a perturbation to the direct operator, as before:

\[
((\sigma_i + \epsilon\delta\sigma_i)A - (L + \epsilon\delta L))(\hat{\chi}_i + \epsilon\delta\hat{\chi}_i) = 0.
\] (29)

At order \(\epsilon\) this is:

\[
(\sigma_iA - L)\epsilon\delta\hat{\chi}_i + (\epsilon\delta\sigma_iA - \epsilon\delta L)\hat{\chi}_i = 0.
\] (30)

Now we pre-multiply by the \(i^{th}\) adjoint eigenvector:

\[
\hat{\xi}^* \cdot (\sigma_iA - L)\epsilon\delta\hat{\chi}_i + \hat{\xi}^* \cdot (\epsilon\delta\sigma_iA - \epsilon\delta L)\hat{\chi}_i = 0.
\] (31)

The first term is zero because of (28). The second term becomes:

\[
\delta\sigma_i\hat{\xi}^* \cdot A \cdot \hat{\chi}_i = \hat{\xi}^* \cdot L\hat{\chi}_i,
\] (32)

which can be rearranged as:

\[
\delta\sigma_i = \frac{\hat{\xi}^* \cdot \delta L\hat{\chi}_i}{\hat{\xi}^* \cdot A \cdot \hat{\chi}_i}.
\] (33)

Both eqns. (27) and (33) show that, once the perturbation operator/matrix is known, we can evaluate the first order eigenvalue drift exactly. To do this, we need to solve one eigenvalue problem to obtain the direct eigenvalue and direct eigenfunction and then another eigenvalue problem to obtain the corresponding adjoint eigenfunction. (Although the eigenvalue is already known for the second eigenvalue problem, it is usually quickest to solve this eigenvalue problem from scratch.) This greatly reduces the number of computations, compared with a finite difference calculation. Equations (27) and (33) are
well known results from perturbation methods (Stewart and Sun, 1990; Hinch, 1991). Although the adjoint equation depends on the choice of the bilinear form, as explained previously, (27) and (33) do not.

When the perturbation operator, \( \delta L \), represents a perturbation to the base state parameters, we label this process a base-state sensitivity analysis. When \( \delta L \) represents a perturbation introduced by additional feedback between the direct variables and the linearized equations, e.g. by a passive feedback device, then we label it a structural sensitivity analysis.

3 A pedagogical example

3.1 The direct governing equations

We will illustrate adjoint sensitivity analysis with a simple example, for which analytical solutions are available. This is a lightly-damped linear oscillator consisting of a mass-spring-damper system, whose displacement, \( x \), obeys the governing equation:

\[
\frac{d^2 x}{dt^2} + b \frac{dx}{dt} + cx = 0,
\]

given some initial conditions. This second order ODE can be written as two first order ODEs by introducing the velocity, \( y \):

\[
\frac{dx}{dt} = y,
\]

\[
\frac{dy}{dt} = -by - cx.
\]

We define the state vector \( q \) and the operator \( L \), which in this case is a matrix of constant coefficients, such that (1) can be written as:

\[
\frac{dq}{dt} - Lq = 0,
\]

where

\[
q = \begin{bmatrix} x \\ y \end{bmatrix}, \quad Lq = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.
\]

3.2 The adjoint governing equations

We define the adjoint operator, \( L^+ \), through (3), which gives

\[
\int_0^T x^+ (y) + y^+ (-by - cx) \, dt = \int_0^T x^+(y)(L^+ q^+)^* x + (L^+ q^+) y \, dt
\]

\[
\Rightarrow \int_0^T x^+ (y)(-c^* y^+) x^* + (x^+ - b^* y^+) y^* \, dt = \int_0^T x^+(y)(L^+ q^+) x^* + (L^+ q^+) y \, dt
\]
Note that here the bilinear form does not involve spatial integration over $V$ because the problem is governed by ODEs. By inspection:
\[
q^+ = \begin{bmatrix} x^+ \\ y^+ \end{bmatrix}, \quad L^+ q^+ = \begin{bmatrix} 0 & -c^* \\ 1 & -b^* \end{bmatrix} \begin{bmatrix} x^+ \\ y^+ \end{bmatrix},
\]
so the adjoint governing equations are:
\[
\frac{dx^+}{dt} = -c^* y^+ , \quad (42) \\
\frac{dy^+}{dt} = -b^* y^+ + x^+. \quad (43)
\]
By comparing with (35),(36) we see that the two first order equations do not create a self-adjoint system\(^3\).

### 3.3 Direct and adjoint eigenmodes

The eigenvalues and eigenvectors of $L$ and $L^+$ can be found by hand calculations:
\[
\sigma_1 = \frac{-b + \sqrt{b^2 - 4c}}{2}, \quad \sigma_1^+ = \sigma_1^*, \quad (44) \\
\hat{q}_1 = \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}_1 = \begin{bmatrix} 2 \\ -b + \sqrt{b^2 - 4c} \end{bmatrix}, \quad \hat{q}_1^+ = \begin{bmatrix} \hat{x}^+ \\ \hat{y}^+ \end{bmatrix}_1 = \begin{bmatrix} -b^* - \sqrt{b^2 - 4c^*} \\ -2c^* - \sqrt{b^2 - 4c^*} \end{bmatrix}. \quad (45)
\]
where the minus sign in front of the square root in the $\hat{y}$ component of $\hat{q}_1^+$ in (45) arises because the system is lightly damped and therefore $b^2 - 4c^*$ is negative.

### 3.4 Sensitivity

We perturb the system (35),(36) with a small feedback mechanism that feeds from $x$ into the first governing equation:
\[
\frac{dx}{dt} = \epsilon x + y, \quad (46) \\
\frac{dy}{dt} = -b y - c x. \quad (47)
\]
Note that we considered $b$ and $c$ to be real, so $b = b^*$ and $c = c^*$. The perturbed state is now:
\[
(L + \delta L)q = \begin{bmatrix} \epsilon & 1 \\ -c & -b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad (48)
\]
or in other words:
\[
\delta L = \begin{bmatrix} \epsilon & 0 \\ 0 & 0 \end{bmatrix}. \quad (49)
\]
\(^3\)Self-adjointness occurs when the direct operator is equal to its adjoint.
We can work out the change in eigenvalue by using the formula derived with the aid of the adjoint eigenfunction (27):

\[ \delta \sigma_1 = \frac{\langle \hat{q}_1^+, \delta L \hat{q}_1 \rangle}{\langle \hat{q}_1^+, \hat{q}_1 \rangle} \]

\[ = \frac{\hat{x}_1^+ \epsilon \hat{x}_1}{\langle \hat{q}_1^+, \hat{q}_1 \rangle} \]

\[ = \epsilon \left( \frac{1}{2} + \frac{b}{2\sqrt{b^2 - 4c}} \right) \]

As a check, we can work out \( \delta \sigma_1 \) by solving exactly the perturbed eigenproblem. We will use the notation \( \sigma_j' = \sigma_j + \delta \sigma_j \) for convenience:

\[ \det \left[ (L + \delta L) - \sigma_j' I \right] = 0, \]

\[ \det \left[ \epsilon - \sigma_j' \begin{bmatrix} 1 & 1 \\ -c & -b - \sigma_j' \end{bmatrix} \right] = 0. \]

\[ (\sigma_j' - \epsilon)(\sigma_j' + b) + c = 0. \]

Therefore:

\[ \sigma_1' = \frac{-(b - \epsilon) + \sqrt{(b - \epsilon)^2 - 4(c - \epsilon b)}}{2} \]

\[ \sigma_2' = \frac{-(b - \epsilon) - \sqrt{(b - \epsilon)^2 - 4(c - \epsilon b)}}{2} \]

To calculate the sensitivity to the perturbation, we differentiate with respect to \( \epsilon \)

\[ \frac{d}{d \epsilon} \left( (b - \epsilon)^2 - 4(c - \epsilon b) \right)^{1/2} = \frac{-(b - \epsilon) + 2b}{(b - \epsilon)^2 - 4(c - \epsilon b)^{1/2}}. \]

So the Taylor expansion of (56) around \( \epsilon = 0 \), at first order, gives:

\[ \sigma_1' = \frac{-b + \sqrt{b^2 - 4c}}{2} + \epsilon \left( \frac{1}{2} + \frac{-(b) + 2b}{2(b^2 - 4c)^{1/2}} \right), \]

and therefore the eigenvalue drift is

\[ \delta \sigma_1 = \epsilon \left( \frac{1}{2} + \frac{b}{2\sqrt{b^2 - 4c}} \right), \]

which is the same as (52), as we wished to show.

### 4 Application to Thermoacoustics

#### 4.1 Reduced order models in thermoacoustics

Fully compressible reacting CFD codes can simulate thermoacoustic oscillations but this is very computationally expensive. Instead, it is much more common to use reduced order
models. In this lecture, we consider a reduced order thermoacoustic model that contains two components. The first component is the acoustic network, which in this case is a tube that is open at both ends. Here, the acoustics is assumed to be one-dimensional but in general 3D acoustics can be modelled by using a Helmholtz solver. The second component is the heat source. This is assumed to be very much smaller than an acoustic wavelength and therefore its dilatation is treated as a compact monopole source of sound for the acoustics. Figure 1 shows a schematic of this reduced order thermoacoustic model. The most important component of the model is the description of how the acoustics affects the heat release. In some models, the heat release is assumed to be an explicit function of the velocity perturbation after a prescribed time delay. In other models, the time evolution of the flame is solved, with input from the acoustic variables, and the heat release is spatially integrated in order to feed back into the acoustic energy equation. Either way, this creates a feedback loop between the acoustics and the flame.

![Figure 1: Reduced-order thermo-acoustic model with acoustically compact flame and mean-flow temperature jump.](image)

### 4.2 Hot wire Rijke tube

The first thermo-acoustic system we will use to demonstrate the adjoint framework is a constant-diameter tube heated by an electrical hot wire or gauze (Matveev, 2003). This is known as a Rijke tube. Full descriptions of such a system, with relevant non-dimensionalizations but no temperature jump across the wire, is given by Balasubramanian and Sujith (2008) and Juniper (2011). One-dimensional acoustic waves occur on top of a mean flow, which undergoes a discontinuity of its uniform properties across the heat source (see figure 1). The mean-flow pressure does not undergo a discontinuity in the low Mach number limit (Dowling, 1995; Nicoud and Wieczorek, 2009). The acoustic momentum, energy equations and heat-release law are, respectively:

\[
\rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0, \quad (61)
\]

\[
\frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \zeta p - \beta q f = -u_c \frac{1}{\gamma} \frac{\partial \gamma}{\partial x} \theta_c, \quad (62)
\]

\[
\dot{q} = \frac{\sqrt{3}}{2} \left[ u_f(t) - \tau \left( \frac{\partial u(t)}{\partial t} \right)_f \right], \quad (63)
\]

where \( u \) and \( p \) are the non-dimensional acoustic velocity and pressure. The heat-transfer coefficient, \( \beta \), is assumed to be constant and its complete expression, encapsulating the
hot wire’s properties and ambient conditions, is reported in Juniper (2011). The acoustic velocity has been non-dimensionalized with the mean flow velocity; the acoustic pressure with $\kappa M_1 \bar{p}$, where $\kappa$ is the heat capacity ratio, $M_1$ is the cold-flow Mach number and $\bar{p}$ the mean-flow pressure; the abscissa with the duct length, $L_a$; the time with $L_a/\bar{c}_1$, where $\bar{c}_1$ is the cold-flow speed of sound. The heat-release rate, $\dot{q}$, is the linearized version of the nonlinear time-delayed law proposed by Heckl (1990), in which the subscript $f$ means that the variable is evaluated at the hot wire’s location $x = x_f$. The time delay between the pressure and heat-release oscillations is modelled by the constant coefficient, $\tau$. This linearization has been performed both in amplitude and time. Eqn. (63) holds providing that $|u_f(t - \tau)| \ll 1$ and $\tau \ll 2/K$, where $K$ is the number of Galerkin modes considered in the numerical discretization (Juniper, 2011; Magri and Juniper, 2013c,b). The non-dimensional density is $\rho = \rho_1$ when $x < x_f$ and $\rho = \rho_2$ when $x > x_f$. The positive mean-flow temperature jump, induced by the heat transferred to the mean-flow, makes the density ratio $\rho_2/\rho_1 < 1$ because of the ideal-gas law.

The system (61), (62) reduces to the D’Alembert equation when $\zeta = 0$ and $\beta_T = 0$

$$\frac{\partial^2 p}{\partial t^2} - \frac{1}{\rho} \frac{\partial^2 p}{\partial x^2} = 0.$$ (66)

The following procedure is applied to find the bases for $u$ and $p$:

1. substitute the decomposition (64) into (66) to find the pressure eigenfunctions $\Psi_j^{(1)}(x), \Psi_j^{(2)}(x)$;
2. substitute the pressure eigenfunctions $\Psi^{(1)}_j(x), \Psi^{(2)}_j(x)$ into the momentum equation (61) to find the velocity eigenfunctions $\Phi^{(1)}_j(x), \Phi^{(2)}_j(x)$;

3. impose the jump condition at the discontinuity, for which $p(x \rightarrow x^-) = p(x \rightarrow x^+)$ and $u(x \rightarrow x^-) = u(x \rightarrow x^+)$, to find the relations between $\alpha^{(1)}_j, \alpha^{(2)}_j, \eta^{(1)}_j, \eta^{(2)}_j$.

Similarly to Zhao (2012), these steps give

$$p(x, t) = \sum_{j=1}^{K} \left\{ -\alpha_j(t) \sin(\omega_j \sqrt{\rho_1} x), \quad 0 \leq x < x_f, \\ -\alpha_j(t) \left( \frac{\sin \gamma_j}{\sin \beta_j} \right) \sin(\omega_j \sqrt{\rho_2}(1 - x)), \quad x_f < x \leq 1, \right. \tag{67}$$

$$u(x, t) = \sum_{j=1}^{K} \left\{ \eta_j(t) \left( \frac{1}{\sqrt{\rho_1}} \right) \cos(\omega_j \sqrt{\rho_1} x), \quad 0 \leq x < x_f, \\ -\eta_j(t) \left( \frac{1}{\sqrt{\rho_2}} \right) \cos(\omega_j \sqrt{\rho_2}(1 - x)), \quad x_f < x \leq 1. \right. \tag{68}$$

where

$$\gamma_j \equiv \omega_j \sqrt{\rho_1} x_f, \quad \beta_j \equiv \omega_j \sqrt{\rho_2}(1 - x_f). \tag{69}$$

Point 3 of the previous procedure provides the equation for the natural acoustic frequencies $\omega_j$

$$\sin \beta_j \cos \gamma_j + \cos \beta_j \sin \gamma_j \sqrt{\frac{\rho_1}{\rho_2}} = 0. \tag{70}$$

The full description and implementation of this method is available in Magri and Juniper (2014) based on the numerical model of Zhao (2012). The continuous adjoint equations of the straight Rijke tube, derived via (7), are

$$\rho \frac{\partial u^+}{\partial t} + \frac{\partial p^+}{\partial x} + \frac{\sqrt{3}}{2} \beta \left( p_f^+ + \tau \left( \frac{\partial p^+}{\partial t} \right)_f \right) \delta_f = 0, \tag{71}$$

$$\frac{\partial u^+}{\partial x} + \frac{\partial p^+}{\partial t} - \zeta p^+ = 0. \tag{72}$$

Note that (71) differs from the adjoint equations presented in previous work (Magri and Juniper, 2013c,a,b) because of the presence of $\rho$, which contains the information about the mean-flow temperature jump. The localized smooth area variation term, $-u_c/\gamma \partial \gamma/\partial x \theta_c$, does not appear in the adjoint equations (71),(72). This is because this term is viewed as a forcing term of the energy equation (62) and the adjoint equations are defined with respect to the homogenous direct equations (see (1),(2),(3)). The direct and conjugate adjoint eigenfunctions are arranged as column vectors $[\hat{u}, \hat{p}]^T$ and $[\hat{u}^{**}, \hat{p}^{**}]^T$, respectively. The structural sensitivity tensor, defined in Magri and Juniper (2013c,b), is

$$S \equiv \frac{\delta \sigma}{\delta C} = \int_0^1 [\hat{u}^{**} \hat{p}^{**}]^T \otimes [\hat{u}, \hat{p}]^T \left( \hat{u} \hat{u}^{**} + \hat{p} \hat{p}^{**} \right) dx + \beta \tau \delta u_f \delta p_f^{**}, \tag{73}$$

where $\otimes$ denotes the dyadic product and $\delta C$ is a matrix of arbitrarily small perturbation coefficients. Each component of this structural perturbation tensor quantifies the effect of a feedback mechanism between a variable and a governing equation, as explained in
4.2 Hot wire Rijke tube

Magri and Juniper (2013c,b). Therefore we can identify the device, and the location, that is most effective at changing the frequency or growth rate of the system just by inspection of the components of the structural sensitivity tensor (73). Here, we discuss the two most significant mechanisms, given by the components

\[ S_{uu} = \frac{\hat{u}^+\hat{u}}{\int_0^1(\hat{u}\hat{u}^+ + \hat{p}\hat{p}^+ + \hat{p}\hat{p}^+ + 1)dx + \beta\tau\hat{u}_f\hat{p}_f^+}, \]  
\[ S_{up} = \frac{\hat{p}^+\hat{u}}{\int_0^1(\hat{u}\hat{u}^+ + \hat{p}\hat{p}^+ + \hat{p}\hat{p}^+ + 1)dx + \beta\tau\hat{u}_f\hat{p}_f^+}. \]  

The reader may refer to Magri and Juniper (2013c,b) for a detailed explanation of the remaining components of the structural sensitivity tensor. The components (74),(75) are shown in fig. 2 as a function of \( x \), which is the location at which the passive device (structural perturbation) acts, both when \( \rho_1/\rho_2 = T_2/T_1 = 1 \) (solid line) and \( \rho_1/\rho_2 = T_2/T_1 = 2 \) (dash-dot line). The component \( S_{uu} \) is the eigenvalue’s sensitivity to a feedback mechanism proportional to the velocity at a given point and affecting the momentum equation. For example, this could be the (linearized) drag force about an obstacle in the flow, as proposed in Magri and Juniper (2013c,b). The real part of \( S_{uu} \) (fig. 2a), being the sensitivity of the system’s growth rate, is positive for all values of \( x \), which means that, whatever value of \( x \) is chosen, the growth rate will decrease if the forcing is in the opposite direction to the velocity, as it is in drag-exerting devices. This tells us that the drag force of a mesh will always stabilize the thermo-acoustic oscillations but is most effective if placed at the ends of the duct. This effect is even stronger if the temperature jump is considered. In summary, this type of feedback greatly affects the growth rate but hardly affects the frequency (fig. 2b), in agreement with Dowling (1995), who performed stability analysis via classical approaches.

The component \( S_{up} \) is the eigenvalue’s sensitivity to a feedback mechanism proportional to the velocity at a given point and affecting the energy equation. This type of feedback hardly affects the growth rate (fig. 2c) but greatly affects the frequency (fig. 2d). By inspection of the linearized heat law (63), we notice that a second hot wire with \( \tau = 0 \) causes this type of feedback, so this analysis shows that it will be relatively ineffective at stabilizing thermo-acoustic oscillations whereas it will be effective at changing the oscillation frequency. A detailed analysis and physical explanation of this finding is reported in Magri and Juniper (2013c,b).

If \( \gamma \neq 0 \), the RHS of eqn. (62) shows that a change in the area can be interpreted as a forcing term, proportional to \(-u_c\), acting on the energy equation. In other words, a positive local smooth change of the cross-sectional area is a feedback mechanism acting like a second hot wire with negative \( \beta \). Hence, the structural sensitivity is provided by \(-S_{up}\). This means that where a control hot wire has a stabilizing effect, a positive change in area in the same location has a destabilizing effect, and vice versa. It is worth mentioning that the structural sensitivity coefficients depicted in figure 2 do not depend on the time delay, \( \tau \), as long as it remains small compared with the oscillation period. We performed calculations for time delays from 0 to 0.03 and observed negligible differences (results not shown).
Figure 2: Two significant components of the structural sensitivity tensor, which quantify the effect of feedback mechanisms, placed at $x$, on the linear growth rate (left frames) and angular frequency (right frames). Solutions with no mean-flow temperature jump (solid lines) and with temperature jump (dash-dotted line). $c_1 = 0.01$, $c_2 = 0.001$, $\tau = 0.01$, $\beta = 0.433$, $x_f = 0.25$, 10 Galerkin modes are used for numerical discretization.

### 4.3 Burke-Schumann flame Rijke tube

The second thermoacoustic system is a Rijke tube heated by a diffusion flame with infinite-rate chemistry, known as a Burke-Schumann flame. The acoustic waves cause perturbations in the velocity field. In turn, these cause perturbations to the mixture fraction, which convect down the flame and cause perturbations in the heat-release rate and the dilatation rate at the flame. The dilatation described above provides a monopole source of sound, which feeds into the acoustic energy. As for the hot wire Rijke tube, we assume that the flame is compact, meaning that the heat release is a point-wise impulsive forcing term for the acoustics. The model and its governing equations are described fully in Magri and Juniper (2014). The governing equations are linearized and the direct and adjoint eigenmodes are calculated.

The real and imaginary components of the least stable direct eigenfunction are shown in figure 3. The real and imaginary parts are in spatial quadrature, which shows that

Figure 3: Real (left) and imaginary (right) components of the mixture fraction perturbation of the least stable eigenfunction for the thermoacoustic system containing a Burke-Schumann flame. The red/blue colour corresponds to positive/negative values. The dashed line is the flame position and only the top half of the flame domain is shown. The acoustic component of the eigenfunction is not shown. The real and imaginary components are in spatial quadrature, showing that this mode contains $z$-perturbations that convect down the flame at almost constant speed.
the mixture fraction perturbation, $\dot{z}$, takes the form of a travelling wave that moves down the flame in the streamwise direction. This shows that a simple model of the flame, in which mixture fraction perturbations convect down the flame at the mean-flow speed and causes heat-release fluctuations when they reach the flame is reasonable. To a first approximation, therefore, the time delay between the velocity perturbation and the subsequent heat-release perturbation scales with $L_f/U$, where $L_f$ is the length of the steady flame and $U$ is the mean-flow speed (which is 1 in this model). The phase delay between the velocity perturbation and the subsequent heat release perturbation therefore scales with $L_f\sigma_i/U$, where $\sigma_i$ is the dominant eigenvalue’s imaginary part, i.e. the linear-oscillation angular frequency.

The absolute value of the corresponding adjoint eigenfunction is shown in figure 4. This is a map, in the flame domain, of the first eigenfunction’s receptivity to species injection. In other words, it shows the most effective regions at which to place an open-loop active device to excite the dominant thermo-acoustic mode. The adjoint eigenfunction has high magnitude around the flame. This is because species injection affects the heat release only if it changes the gradient of $\dot{z}$ at the flame itself, which is achieved by injecting species around the flame. Its magnitude increases towards the tip of the flame, where $\nabla \dot{z}$ is weakest. This is because mixture fraction fluctuations diffuse out as they convect down the flame, which means that open-loop forcing has a proportionately large influence on the mixture fraction towards the tip. From a practical point of view, this shows that open-loop control of the mixture fraction has little influence at the injection plane but great influence at the flame tip. In this case, this could be achieved by injecting species at the wall.

Figure 5 shows two of the base state sensitivities for this model: to changes in the stoichiometric mixture fraction (top) and to changes in the flame aspect ratio (bottom). These results, obtained by the adjoint-based approach, have been checked against the solutions obtained via finite difference and agree to within $\sim O(10^{-9})$. These figures are useful from a design point of view. For example, they reveal that, at $Z_{sto} = 0.15$ and $\alpha = 0.35$, changes in $Z_{sto}$ strongly influence the growth rate but that, at $Z_{sto} = 0.15$ and $\alpha = 0.30$, changes in $Z_{sto}$ strongly influence the frequency instead. This demonstrates an inconvenient feature of thermo-acoustic instability: it is exceedingly sensitive to small changes in the operating point. In this analysis, the length of the unperturbed flame, $L_f$, emerges as a key parameter. This is defined here as the distance between the inlet and the tip of the steady flame and is shown by the black lines in figure 5. Lines of constant $\delta \sigma / \delta Z_{sto}$ and $\delta \sigma / \delta \alpha$ very nearly follow the lines of constant $L_f$. This can be explained physically by considering the physical mechanism behind thermoacoustic insta-
Figure 5: Sensitivity of the growth rate, $\sigma_r$, (left frames) and frequency, $\sigma_i$ (right frames) to changes in the stoichiometric mixture fraction, $Z_{sto}$, (top frames) and flame aspect ratio, $\alpha$, (bottom frames). These sensitivities are shown as functions of $Z_{sto}$ (vertical axis, from 0.025 to 0.20) and $\alpha$ (horizontal axis, from 0.25 to 0.4). The contour levels show: Real($\partial\sigma/\partial Z_{sto}$) on the top left, Imag($\partial\sigma/\partial Z_{sto}$) on the top right, Real($\partial\sigma/\partial \alpha$) on the bottom left, Imag($\partial\sigma/\partial \alpha$) on the bottom right. These are two of the base state sensitivities for this model. The sensitivities to the other flame parameters and to the acoustics are also calculated but are not shown. The numbered black lines are contours of constant flame length.

...ability: acoustic velocity perturbations cause mixture fraction perturbations at the base of the flame. These are convected downstream and cause a heat-release perturbation some time later. The time delay between acoustic velocity and subsequent heat release, $\tau$, scales with $L_f/U$, where $L_f$ is the length of the flame. The influence of this heat-release perturbation on the growth rate or the frequency of the acoustic wave depends on the phase of the heat release relative to the phase of the velocity or pressure, which are in temporal quadrature. This is why the base-state sensitivity plots are in spatial quadrature in parameter space. The phase delay, $\psi$, is given by $\tau/T$, where $T = 2\pi/\sigma_i$. In this simple model, $\delta\sigma$ depends only on $\psi$, which means that the eigenvalue drifts in figure 5 should collapse onto a single line when plotted as a function of $L_f\sigma_i/U$. This is shown in figure 6 for $\delta\sigma/\delta Z_{sto}$ (top) and $\delta\sigma/\delta \alpha$ (bottom). The data collapse reasonably closely to a line, showing that the simple model provides a physical understanding of the sensitivities, while figure 5 provides their exact values. Although not shown here, once the direct and adjoint eigenfunctions have been calculated, all the other base state sensitivities follow at very little extra cost, demonstrating the utility of these techniques.
Figure 6: Sensitivity of the growth rate, $\sigma_r$, (left frames) and frequency, $\sigma_i$, (right frames) to changes in the stoichiometric mixture fraction, $Z_{sto}$, (top frames) and flame aspect ratio, $\alpha$, (bottom frames). These sensitivities are shown as functions of $L_f \sigma_i/U$ (horizontal axis), which is plotted from 1.6 to 5.6. The data, which is the same as that in figure 5, nearly collapses to a single line in each figure.
5 Concluding remarks

The aim of this lecture is to show how adjoint sensitivity analysis can be applied to thermoacoustics. We describe the physical meaning of the adjoint eigenfunction in terms of the system’s receptivity to open-loop forcing and show how to combine the direct and adjoint eigenfunctions in order to obtain an analytical formula for the first-order eigenvalue drift.

The results in this paper are for a simple thermo-acoustic model and are, of course, only as accurate as the model itself. The adjoint-based techniques, however, can readily be applied to more realistic models, as long as they can be linearized. This could quickly reveal, for example, the best position for an acoustic damper in a complex acoustic network, the optimal change in the flame shape and the best strategies for open loop control. The usefulness of adjoint techniques applied to thermo-acoustics is that, in a few calculations, one can predict accurately how the growth rate and frequency of thermo-acoustic oscillations are affected either by all possible passive control elements in the system or by all possible changes to its base state.

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