Information-Theoretic Results on Communication Problems with Feed-forward and Feedback

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Oral Defense

Dept. of EECS, University of Michigan

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Thesis Outline

1. Source Coding with Feed-forward
   - Point to Point Source Coding (Chap. 2)
     - Problem Statement
     - Rate-Distortion Function
     - Error Exponents
   - Computation (Chap. 4)
   - Multiple Descriptions (Chap. 5)

2. Channel Coding with Feedback (Chap. 3,4)
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     - Problem Statement
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   - Computation (Chap. 4)
   - Multiple Descriptions (Chap. 5)

2. Channel Coding with Feedback (Chap. 3, 4)
Lossy Data Compression

- Source $X$, reconstruction $\hat{X}$
- Distortion measure $d(X, \hat{X})$

Rate-distortion function

Minimum rate $R$ for distortion level $D$ (Shannon '59)
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Source Coding with Side-Information

- $X, Y$ correlated random variables
- *Example*: Temperature at nearby cities
- Presence of $Y \Rightarrow$ lower rate for given distortion $D$

Rate-distortion function $R_{WZ}(D)$ [Wyner, Ziv ’76]
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Example: Block length $N = 5$

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### Source Coding with Side-Information

**Diagram:**
- **Encoder** with input $X$ and output $Rate R$.
- **Decoder** with input $Y$ and output $\hat{X}$.

**Example:** Block length $N = 5$

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Suppose there is a delay in the side info available at the decoder.
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Time  
1  2  3  4  5  6  7  8  9  10

Source  
\(X_1\)  \(X_2\)  \(X_3\)  \(X_4\)  \(X_5\)  \(X_6\)  \(X_7\)  \(X_8\)  \(X_9\)  \(X_{10}\)

Encoder  
-  -  -  -  -  \(W\)  -  -  -  -  \(W\)

Side Info  
-  -  -  \(Y_1\)  \(Y_2\)  \(Y_3\)  \(Y_4\)  \(Y_5\)  \(Y_6\)  \(Y_7\)

Decoder  
\(\hat{X}_1\)  \(\hat{X}_2\)  \(\hat{X}_3\)  \(\hat{X}_4\)  \(\hat{X}_5\)
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What is feed-forward?

The source field itself available with delay at decoder.

Here, block length = 5, delay = 6 time units.
What is feed-forward?

The source field itself available with delay at decoder.

![Diagram showing encoder and decoder with rate R and slow noiseless path]

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Encoder - - - - - W - - - - - W
Extra info - - - - - - - X$\_1$ $X_2$ $X_3$ $X_4$
Decoder $\hat{X}\_1$ $\hat{X}_2$ $\hat{X}_3$ $\hat{X}_4$ $\hat{X}_5$

Here, block length = 5, delay = 6 time units.
Feed-forward ⇒ Decoder knows some of the past source samples.

FF with delay $k$, block length $N$.

To produce $\hat{X}_n$, the decoder knows index $W$ and $(X_1, \ldots, X_{n-k})$. 
Stock Market Example

- Behavior of a particular stock over an $N$-day period.
- Stock price - modeled as a $k + 1$-state Markov chain.
- Value of stock: Markov source $\{X_n\}$
Insider- *a priori* knowledge about behavior of stock

Investor has stock for $N$ days, needs to know when value drops.

Insider: gives information to investor at rate $R$. 
Insider and Investor

Reconstruction

- Decision of investor on day $n$: $\hat{X}_n$
  - $\hat{X}_n = 1 \Rightarrow$ price drop from day $n - 1$ to $n$
  - Otherwise $\hat{X}_n = 0$
  - Distortion $= 1$ if $\hat{X}_n$ is wrong

$X_1 \ldots X_N$
Insider and Investor

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Feed-forward!

- Before day $n$, investor knows previous stock values $X^{n-1}$, makes decision $\hat{X}_n$
- Minimum info (in bits/sample) needed to predict drops with distortion $D$?
Feed-Forward: A Formal Definition

Previous: [Weissman et al 03], [Pradhan 04], [Martinian et al 04]

- **Source** $X$: Alphabet $\mathcal{X}$, reconstruction alphabet $\hat{\mathcal{X}}$
- **Encoder**: Rate $R$, $e: \mathcal{X}^N \to \{1, \ldots, 2^{NR}\}$
- **Decoder**: knows all the past $(n-k)$ source samples to reconstruct $n$th sample.

$$g_n: \{1, \ldots, 2^{NR}\} \times \mathcal{X}^{n-k} \to \hat{\mathcal{X}}, \quad n = 1, \ldots, N.$$
Distortion measure $d_N(X^N, \hat{X}^N)$.

**GOAL**

Given any source $X$, find the least $R$ such that

$$E[d_N(x^N, \hat{x}^N)] \leq D.$$
Inspired by [Marko ’73]: work on bidirectional communication

[Massey ’90] The directed information flowing from $A^N$ to $B^N$

$$I(A^N \rightarrow B^N) = \sum_{n=1}^{N} I(A^n; B_n | B^{n-1}).$$

Interestingly:

$$I(A^N \rightarrow B^N) = I(A^N; B^N) - \sum_{n=2}^{N} I(B^{n-1}; A_n | A^{n-1})$$

$$I(A^N \rightarrow B^N) = I(A^N; B^N) - I(0B^{N-1} \rightarrow A^N)$$
Directed Information

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Causality in Information Flow

\[ I(A^N; B^N) = I(A^N \rightarrow B^N) + I(0B^{N-1} \rightarrow A^N) \]

- \( I(A^N \rightarrow B^N) \): How causal knowledge of \( A_n \)'s reduces the uncertainty in \( B_n \)
- Example: GNP vs Money Supply [Geweke '82]
Without FF, need $I(\hat{X}^N; X^N)$ bits to represent $X^N$ with $\hat{X}^N$.

- With feed-forward, to produce $\hat{X}_n$, the decoder knows $X^{n-1}$.
- Number of bits required is reduced by $I(\hat{X}_n; X^{n-1} | \hat{X}^{n-1})$. 
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With delay 1 feed-forward, we need

\[ I(\hat{X}^N; X^N) - \sum_{n=2}^{N} I(\hat{X}_n; X^{n-1} | \hat{X}^{n-1}) \text{ bits.} \]

Directed information from \( \hat{X}^N \) to \( X^N \)!
Delay $k$ feed-forward

With delay $k$ feed-forward, to produce $\hat{X}_n$, the decoder knows $X^{n-k}$.

No. of bits: $I(\hat{X}^N; X^N) - \sum_{n=k+1}^N I(\hat{X}_n; X^{n-k} | \hat{X}^{n-1})$

Not Directed Information- will denote it $I_k(\hat{X}^N \rightarrow X^N)$
  - ‘$k$–directed information’.

Rate-distortion function

- Optimize $I_k$ over $P_{\hat{X}^N | X^N}$ that satisfy distortion

- No savings for discrete memoryless source w/ single-letter distortion measure
Delay $k$ feed-forward

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**Rate-distortion function**

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General source, general distortion measure

- Even when source is stationary, ergodic:
  - with feed-forward, optimal joint distribution may not be.

- Source could be non-stationary, non-ergodic
- Sequence of distortion functions $d_n(.,.)$

- Need information-spectrum methods [Han, Verdu '93, '95]
General source, general distortion measure

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Definitions

\( a_1, a_2, \ldots : \) random sequence

- \( \lim \sup_{\text{in} \; \text{prob}} a_n = \overline{a} \): Smallest number \( \alpha \) such that
  \[
  \lim_{n \to \infty} \Pr(a_n > \alpha) = 0.
  \]

- We will need
  \[
  i_k(\hat{x}^n \to x^n) = \frac{1}{n} \log \frac{P(x^n, \hat{x}^n)}{P(x^n) \cdot \prod_{i=1}^{n} P(\hat{x}_i | \hat{x}_i^{-1}, x_i^{-k})}
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Definitions

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- pdf
Rate-Distortion Theorem for a general source

- \( P_X = \{P_{X_1}, P_{X^2}, \ldots, P_{X^N}, \ldots\} \)

- \( P_{\hat{X}|X} = \{P_{\hat{X}_1|X_1}, P_{\hat{X}_2|X^2}, \ldots, P_{\hat{X}_N|X^N}, \ldots\} \)

**Theorem**

For arbitrary source \( X \) with distribution \( P_X \), the rate-distortion function with feed-forward delay \( k \), the infimum of all achievable rates at probability-1 distortion \( D \), is given by

\[
R_{ff}(D) = \inf_{P_{\hat{X}|X}: \rho(P_{\hat{X}|X}) \leq D} \bar{I}_k(\hat{X} \rightarrow X),
\]

where

\[
\rho(P_{\hat{X}|X}) \triangleq \limsup_{\text{inprob}} d_n(x^n, \hat{x}^n)
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Rate-Distortion Theorem for a general source

Theorem

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The story so far . . .

- What is feed-forward in source coding?
- Directed information and why it occurs
- Feed-forward rate-distortion result for general sources, distortions

Next . . .

How to compute the rate-distortion function?
The story so far . . .

- What is feed-forward in source coding?
- Directed information and why it occurs
- Feed-forward rate-distortion result for general sources, distortions

Next . . .

How to compute the rate-distortion function?
Source X with distribution $P_X$.
Find $P_{\hat{X}|X}$ to minimize $\bar{T}_k(\hat{X} \rightarrow X)$ s.t

$$\limsup_{n \to \infty} d_n(X^n, \hat{X}^n) \leq D$$

in prob

Multi-letter optimization - difficult!
Source $X$ with distribution $P_X$.

Find $P_{\hat{X}|X}$ to minimize $\overline{I}_k(\hat{X} \rightarrow X)$ s.t

$$\limsup_{n \to \infty} d_n(X^n, \hat{X}^n) \leq D \text{ in prob}$$

Multi-letter optimization - difficult!
Pick a conditional distribution $P_{\hat{X}|X} = \{P_{\hat{X}_n|X_n}\}$

For what sequence of distortion measures $d_n$ does $P_{\hat{X}|X}$ achieve the infimum in the rate-distortion formula?

Approach—similar in spirit to [Csiszar and Korner]
A stationary, ergodic source $X$ characterized by $P_X = \{ P_X^n \}_{n=1}^{\infty}$ with feed-forward delay $k$. $P_{\hat{X}|X} = \{ P_{X^n|X^n} \}_{n=1}^{\infty}$ is a conditional distribution such that the joint distribution is stationary and ergodic. Then $P_{\hat{X}|X}$ achieves the rate-distortion function if for all sufficiently large $n$, the distortion measure satisfies

$$d_n(x^n, \hat{x}^n) = -c \cdot \frac{1}{n} \log \frac{P_{X^n, \hat{X}^n}(x^n, \hat{x}^n)}{\bar{P}^k_{\hat{X}^n|X^n}(\hat{x}^n|x^n)} + d_0(x^n),$$

where

$$\bar{P}^k_{\hat{X}^n|X^n}(\hat{x}^n|x^n) = \prod_{i=1}^{n} P_{\hat{X}_i|X^{i-k}, \hat{X}^{i-1}}(\hat{x}_i|x^{i-k}, \hat{x}^{i-1})$$

and $c$ is any positive number and $d_0(.)$ is an arbitrary function.
Stock example revisited

- Value of the stock: Markov source \( \{X_n\} \)

- Decision of investor on day \( n \): \( \hat{X}_n \) (0 or 1)
Stock example revisited

Value of the stock: Markov source \( \{X_n\} \)

Decision of investor on day \( n \): \( \hat{X}_n \) (0 or 1)
- $R_{ff}(D)$: Minimum rate the investor needs to predict drops with distortion $D$.

- Try first-order Markov joint distribution
- Distortion can be cast in the required form!
Proposition

The minimum rate in \((\text{bits/sample})\) is

\[
\sum_{i=1}^{k-1} \pi_i \left[ h(p_i, q_i, 1 - p_i - q_i) - h(\epsilon, 1 - \epsilon) \right] 
+ \pi_k \left( h(q_k, 1 - q_k) - h(\epsilon, 1 - \epsilon) \right)
\]

where

- \(h()\) is the entropy function
- \(\left[ \pi_0, \pi_1, \cdots, \pi_k \right]\) is the stationary distribution of the stock
- \(\epsilon = \frac{D}{1 - \pi_0}\)
Computing Rate-distortion function with FF

1. ‘Predict’ a conditional distribution
2. Check if distortion function can be put into required form.

Next . . . A multi-terminal problem
Computing Rate-distortion function with FF

1. ‘Predict’ a conditional distribution
2. Check if distortion function can be put into required form.

Next . . . A multi-terminal problem
- Source $X$: compressed into packets
- Packets may be dropped
- Compress $X$ into two different packets
Multiple Descriptions

- Source $X$: compressed into packets
- Packets may be dropped
- Compress $X$ into two *different* packets
Multiple Descriptions

- Rate $R_1$ yields reconstruction $\hat{X}_1$ with distortion $D_1$
- Rate $R_2$ yields reconstruction $\hat{X}_2$ with distortion $D_2$

- Want better quality $D_0$ if both packets received
- $\hat{X}_1$ and $\hat{X}_2$ need to refine each other!
Goal

Given i.i.d source $P_X$:

Find all achievable $(R_1, R_2, D_1, D_2, D_0)$

- Still an open problem
- Studied by [Cover, El Gamal], [Ahlswede], [Zhang, Berger], ...
- Best known rate-region: [Zhang, Berger ’87]
Rate | Distortion
---|---
$R_1$ bits/sample | $D_1$
$R_2$ bits/sample | $D_2$
$R_1 + R_2$ bits/sample | $D_0$

Still an open problem

Studied by [Cover, El Gamal], [Ahlswede], [Zhang, Berger], . . .

Best known rate-region: [Zhang, Berger '87]
Source propagates to destination with delay

To reconstruct $\hat{X}_n$, decoder has packet and $X^{n-k}$
Alice has an i.i.d binary source $\sim$ Bernoulli(1/2)

Bob and Carol: distortion $d$ using $R_B, R_C$ bits/sample

Dave gets Bob’s bits and Carol’s bits- needs to reconstruct perfectly!

Characterize

$r_{sum}(d) \triangleq$ Smallest sum-rate $R_B + R_C$ for distortion $(d, d, 0)$
Same model as before, one extra feature...

- After Carol reconstructs each sample, Alice reveals the value to her.
  *Feed-forward*

- Before reconstructing each sample, Carol knows past source samples

Characterize with feed-forward

\[ r_{\text{sum}}(d) \triangleq \text{Smallest sum-rate } R_B + R_C \text{ for } (d, d, 0) \]
Same model as before, one extra feature...

- After Carol reconstructs each sample, Alice reveals the value to her. *Feed-forward*
- Before reconstructing each sample, Carol knows *past* source samples

Characterize with feed-forward

\[ r_{\text{sum}}(d) \triangleq \text{Smallest sum-rate } R_B + R_C \text{ for } (d, d, 0) \]
Encoder mappings: $e_m : \mathcal{X}^N \rightarrow \{1, \ldots, 2^{NR_m}\}, \quad m = 1, 2.$

Mappings for decoders 1 and 0:

$$
g_1 : \{1, \ldots, 2^{NR_1}\} \rightarrow \hat{\mathcal{X}}_1^N
$$

$$
g_0 : \{1, \ldots, 2^{NR_1}\} \times \{1, \ldots, 2^{NR_2}\} \rightarrow \hat{\mathcal{X}}_0^N
$$

A sequence of mappings for decoder 2:

$$
g_{2n} : \{1, \ldots, 2^{NR_2}\} \times \mathcal{X}^{n-k} \rightarrow \hat{\mathcal{X}}_2, \quad n = 1, \ldots, N.
$$
Encoder mappings: $e_m : \mathcal{X}^N \rightarrow \{1, \ldots, 2^{NR_m}\}, \quad m = 1, 2$.

Mappings for decoders 1 and 0:

$$g_1 : \{1, \ldots, 2^{NR_1}\} \rightarrow \hat{\mathcal{X}}_1^N$$
$$g_0 : \{1, \ldots, 2^{NR_1}\} \times \{1, \ldots, 2^{NR_2}\} \rightarrow \hat{\mathcal{X}}_0^N$$

A sequence of mappings for decoder 2:

$$g_{2n} : \{1, \ldots, 2^{NR_2}\} \times \mathcal{X}^{n-k} \rightarrow \hat{\mathcal{X}}_2, \quad n = 1, \ldots, N.$$
Zhang-Berger '87

Let $P_{U, \hat{X}_1, \hat{X}_2, \hat{X}_0 | X}$ be such that

\begin{align*}
    E_{d_m}(X; \hat{X}_m) &\leq D_m, \quad m = 0, 1, 2 \\
    R_1 &> I(X; \hat{X}_1 U) \quad R_2 > I(X; \hat{X}_2 U) \\
    R_1 + R_2 &> I(X; \hat{X}_1 U) + I(X; \hat{X}_2 U) + I(X; \hat{X}_0 | \hat{X}_1 \hat{X}_2 U) \\
    &\quad + I(\hat{X}_1; \hat{X}_2 | X U)
\end{align*}
Stringent Decoder 0 distortion ⇒ Need correlation in $\hat{X}_1, \hat{X}_2$

‘Cloud’ Center
- $U$: Cloud center of $X$ sent to all decoders
- Rate $I(U; X)$ each for decoder 1 and 2

Penalty Term
- Can’t have $\hat{X}_1 \sim P(\hat{X}_1|XU)$ and $\hat{X}_2 \sim P(\hat{X}_2|XU)$ indep’ly
- Need to be jointly distributed: $\sim P(\hat{X}_1, \hat{X}_2|XU)$
- $I(\hat{X}_1; \hat{X}_2|XU)$: Penalty in sum-rate

FF: decoder 2 knows past samples with some delay
- Can help build correlation!
Stringent Decoder 0 distortion $\Rightarrow$ Need correlation in $\hat{X}_1, \hat{X}_2$

‘Cloud’ Center

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- FF: decoder 2 knows past samples with some delay
- Can help build correlation!
Coding Strategy

- Consider $B$ long blocks of source samples

  \[ X_1 \ldots X_N \quad X_{N+1} \ldots X_{2N} \quad \ldots \ldots \quad X_{NB} \]

- While encoding one block, give ‘preview’ of next block

![Diagram showing blocks and preview](image-url)
Coding Strategy

Restricted encoding for user 1- within cell $j$

After reconstructing block $b$:
- User 1 gets 'preview' of block $b + 1$
- User 2 knows it too- due to FF!

Block-Markov, superposition; [Cover,Leung], [Willems] for MAC
Coding Strategy

Restricted encoding for user 1- within cell $j$

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Block-Markov, superposition; [Cover,Leung], [Willems] for MAC
Rate region with FF

Theorem

\[ P_{U, \hat{X}_1, \hat{X}_2, \hat{X}_0 | X} \] such that

\[ E_{d_m}(X; \hat{X}_m) \leq D_m, \quad m = 0, 1, 2 \]

\[ R_1 > I(X; \hat{X}_1 U) \]

\[ R_2 > I(X; \hat{X}_2 | U) + \max\{0, R_1 - I(X \hat{X}_2; \hat{X}_1 | U)\} \]

\[ R_1 + R_2 > I(X; \hat{X}_1 U) + I(X; \hat{X}_2 | U) + I(X; \hat{X}_0 | \hat{X}_1 \hat{X}_2 U) \]

\[ + I(\hat{X}_1; \hat{X}_2 | XU) + \max\{0, R_1 - I(X \hat{X}_2; \hat{X}_1 | U)\} \]
Improvement over [Zhang-Berger]

- Fix $R_1 = I(X; \hat{X}_1 U) + \epsilon$
- If $\max\{0, R_1 - I(X \hat{X}_2; \hat{X}_1 | U)\} = 0$:
  - Savings in $R_2 = I(U; X)$ bits/sample
  - FF conveys ‘cloud center’ $U$ for free
- If $\max\{0, R_1 - I(X \hat{X}_2; \hat{X}_1 | U)\} \neq 0$:
  - Savings in $R_2 = I(\hat{X}_1; \hat{X}_2 | XU)$ bits/sample
  - FF eliminates penalty term
Fix $R_1 = I(X; \hat{X}_1 U) + \epsilon$

If $\max\{0, R_1 - I(X\hat{X}_2; \hat{X}_1 | U)\} = 0$:
- Savings in $R_2 = I(U; X)$ bits/sample
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If $\max\{0, R_1 - I(X\hat{X}_2; \hat{X}_1 | U)\} \neq 0$:
- Savings in $R_2 = I(\hat{X}_1; \hat{X}_2 | XU)$ bits/sample
- FF eliminates penalty term
Example: No feed-forward

Without FF [Zhang Berger ’87]

\[ r_{sum}(d) \geq 2 - h \left( \frac{4d + 1 - \sqrt{12d^2 - 4d + 1}}{2} \right) \]
- (a): Lower bound without feed-forward [Zhang, Berger ’87]
- (b): Achievable sum-rate with FF- better than optimal w/o FF
- (c): Rate-dist lower bound with FF- $R_B + R_C > 2(1 - h(d))$
(a): Lower bound without feed-forward [Zhang, Berger ’87]
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(c): Rate-dist lower bound with FF- $R_B + R_C > 2(1 - h(d))$
Summary

- Feed-forward: helps build correlation
- Single-letter achievable region for MD with FF
- FF helps even for an i.i.d source

- How to use feed-forward to all decoders?
- FF for Gaussian multiple descriptions [Pradhan IT ’07]
Summary

- Source Coding with feed-forward
  - Directed Information
  - Rate-Distortion Function
  - How to evaluate optimization
  - Multiple Descriptions with FF

- Channel Coding with feedback

Some questions...

- Feedback/FF in multi-terminal setting
- Noisy feedback/FF
- Applications of directed-info like quantities
Summary

- Source Coding with feed-forward
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