Achievable Rates for the Broadcast Channel with Feedback

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Broadcast Channel

$P(Y_1, Y_2|X)$

Feedback

$x_n$: function of $(Y_1, 1, \ldots, Y_1, Y_1, \ldots, Y_1)$ and $(Y_2, 1, \ldots, Y_2, Y_2, \ldots, Y_2)$

Does feedback help? Yes!
Broadcast Channel

Feedback

- $X_n$: function of $(Y_{11}, \ldots, Y_{1n-1})$ and $(Y_{21}, \ldots, Y_{2n-1})$
- Does feedback help? Yes!
Example [Dueck ’80]

Channel

$X = (X_0, X_1, X_2)$

$Y_1 = (X_0, X_1 \oplus Z)$

$Y_2 = (X_0, X_2 \oplus Z)$

$X_0, X_1, X_2$ & noise $Z$ binary

$P(Z = 0) = P(Z = 1) = \frac{1}{2}$

No feedback

- 1 common bit/channel use through ‘clean’ input $X_0$
- $X_1, X_2$ are useless
With feedback

\[ Y_1 = (X_0, X_1 \oplus Z) \]

\[ Y_2 = (X_0, X_2 \oplus Z) \]

- Send 1 separate bit to Rx each through \( X_1, X_2 \)
- Rx 1 receives \( X_1 \oplus Z \), Rx 2 receives \( X_2 \oplus Z \)
- In next transmission, set \( X_0 = Z \)
- Both can now recover their bits \( \Rightarrow 2 \) bits/channel use!
Prior work

- FB does not improve capacity region for a physically degraded BC - [El Gamal ’80]
- Binary example - [Dueck ’80]
- Scalar Gaussian BC - [Ozarow, Leung ’84]
- Scalar Gaussian BC - [Bhaskaran ’08] (partial feedback)
**Prior work**

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**Our result**

Achievable rate region for discrete memoryless BC with feedback

- Initially transmit at high rates
- Resolution information is correlated
- Use *joint* source-channel coding to resolve
Broadcast without feedback

Marton rate-region

- **Encoding**: \( \text{# codewords in each product bin} \sim 2^{nI(U;V)} \)
- **Decoding**: \(|\text{Codebook 1}| < 2^{nI(U;Y_1)}\), \(|\text{Codebook 2}| < 2^{nI(V;Y_2)}\)
Outside Marton’s region

Suppose:

- # codewords in each product bin $\sim 2^{nI(U;V)}$, but ...
- $|\text{Codebook 1}| > 2^{nI(U;Y_1)}$, $|\text{Codebook 2}| > 2^{nI(V;Y_2)}$
  - Receivers cannot decode

- Rx 1: List of $U$ typical with $Y_1$
- Rx 2: List of $V$ typical with $Y_2$

Want to resolve these lists in next block

What information should we send?
Lists are correlated

\[ UV \rightarrow X \rightarrow (Y_1, Y_2) \]

Gaussian \( U, V \):
Lists are correlated

\[ UV \rightarrow X \rightarrow (Y_1, Y_2) \]

Gaussian \( U, V \):
Lists: Realization of this pair of correlated sources
Transmit these correlated sources efficiently in the next block
Correlated sources over a BC

Correlated sources $S_1, S_2$, codebooks of $A$ and $B$

$A \sim P_{A|S_1}$

$B \sim P_{B|S_2}$

$2^{nH(S_1)}$ bins

$2^{nH(S_2)}$ bins

- The codebooks are correlated $\Rightarrow$ bins can be smaller
- Can have more bins in $A$ and $B$ codebooks of a given size
- Decoding: $|\text{Codebook 1}| < 2^{nl(S_1A;Y_1)}$, $|\text{Codebook 2}| < 2^{nl(S_2B;Y_2)}$
First quantize \((S_1, S_2)\) to common \(C\) codebook at rate \(\rho_0\)

\[
A \sim P_{A|S_1}C
\]

\(2^{nH(S_1)}\) bins

\[
2^{n\rho_1}
\]

\[
B \sim P_{B|S_2}C
\]

\(2^{nH(S_2)}\) bins

\[
2^{n\rho_2}
\]

**Encoding:** \(\rho_0, \rho_1, \rho_2\) large enough to cover all \(S, T\) pairs

**Decoding:** \(H(S_1) + \rho_0 + \rho_1\) small enough to decode \(A, C\)

\(- H(S_2) + \rho_0 + \rho_2\) need to be small enough to decode \(B, C\)

Eliminate \(\rho_0, \rho_1, \rho_2\)
Block-Markov Coding

Source 1: $UY_1$
Source 2: $VY_2$

$A, B, C$ to cover the sources

$(U, Y_1)_b$  $(V, Y_2)_b$

$(A, B, C)_{b+1}$

$Y_{1b+1}$  $Y_{2b+1}$

$(A, B, C)_{b+2}$

Dependence propagates across blocks!
Want a *stationary* distribution of sequences in each block

\[ P_{UV} \cdot P_{ABC} \cdot P_{X|UV\ ABC} \cdot P_{Y|X_1X_2} \]

How to ensure this?
Stationarity

Want a *stationary* distribution of sequences in each block

\[ P_{UV} \cdot P_{ABC} \cdot P_{X|UV \ ABC} \cdot P_{Y|X_1 X_2} \]

How to ensure this?

- Previous block: \( \tilde{U}, \tilde{V}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{Y}_1, \tilde{Y}_2 \)
- Generate \((A, B, C)\) for next block \( \sim Q_{ABC|\tilde{U}\tilde{V}\tilde{A}\tilde{B}\tilde{C}\tilde{Y}_1 \tilde{Y}_2} \)

Consistency

\( A_{b+1}, B_{b+1}, C_{b+1} \) are \( \sim P_{ABC} \) if

\[
P(A, B, C) = \sum_{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{u}, \tilde{v}, \tilde{y}_1, \tilde{y}_2} Q(A, B, C|\tilde{a}, \tilde{b}, \tilde{c}, \tilde{u}, \tilde{v}, \tilde{y}_1, \tilde{y}_2) \cdot P(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{u}, \tilde{v}, \tilde{y}_1, \tilde{y}_2)
\]
Block-Markov Coding

- Fix $P_{ABC} \cdot P_{UV} \cdot P_{X|ABCUV} \cdot P_{Y_1Y_2|X}$
- The joint distribution in two successive blocks is then

\[
P \triangleq P_{\tilde{A}\tilde{B}\tilde{C}\tilde{U}\tilde{V}\tilde{X}\tilde{Y}_1\tilde{Y}_2} \cdot Q_{ABC|\tilde{A}\tilde{B}\tilde{C}\tilde{U}\tilde{V}\tilde{Y}_1\tilde{Y}_2} \cdot P_{ABCUVXY_1Y_2}
\]
Achievable Rate Region

**Theorem**

For the BC $P_{Y|X_1X_2}$, fix any joint distribution

$$P_{ABC} \cdot P_{UV} \cdot P_{X|ABCUV} \cdot P_{Y_1Y_2|X}$$

and $Q_{ABC|\tilde{A}\tilde{B}\tilde{C}\tilde{U}\tilde{V}Y_1Y_2}$ that is consistent. Then the following rate-region is achievable.

\[
\begin{align*}
R_1 &< I(U; Y_1|AC) + I(\tilde{U}AC; Y_1|\tilde{A}\tilde{C}\tilde{Y}_1) - I(\tilde{V}\tilde{B}\tilde{Y}_2; AC|\tilde{U}\tilde{A}\tilde{C}\tilde{Y}_1) \\
R_2 &< I(V; Y_2|BC) + I(\tilde{V}BC; Y_2|\tilde{B}\tilde{C}\tilde{Y}_2) - I(\tilde{U}\tilde{A}\tilde{Y}_2; BC|\tilde{V}\tilde{B}\tilde{C}\tilde{Y}_2) \\
R_1 + R_2 &< \min\{T_1, T_2, T_3\} + I(U; Y_1|AC) + I(\tilde{U}A; Y_1|\tilde{A}\tilde{C}\tilde{Y}_1C) \\
&- I(\tilde{U}\tilde{A}\tilde{Y}_1; B|\tilde{V}\tilde{B}\tilde{C}\tilde{Y}_2C) + I(V; Y_2|BC) + I(\tilde{V}B; Y_2|\tilde{B}\tilde{C}\tilde{Y}_2C) \\
&- I(\tilde{V}\tilde{B}\tilde{Y}_2; A|\tilde{U}\tilde{A}\tilde{C}\tilde{Y}_1C) - I(A; B|\tilde{U}\tilde{V}\tilde{A}\tilde{B}\tilde{C}\tilde{Y}_1\tilde{Y}_2C) - I(U; V)
\end{align*}
\]
Scalar Gaussian BC

\[ Y_1 = X + N_1 \]
\[ Y_2 = X + N_2 \]

Input Power \( P \), \( N_1, N_2 \) iid \( \sim \mathcal{N}(0, \sigma^2) \)

\[ C_{no-FB} = \frac{1}{2} \log_2(1 + \frac{P}{\sigma^2}) \]

Achieve through Dirty Paper Coding

\[ V = \sqrt{\alpha P} Q_1, \quad U = \sqrt{\alpha P} Q_2 + \frac{\alpha P}{\alpha P + \sigma^2} V, \quad Q_1, Q_2 \text{ iid } \sim \mathcal{N}(0, 1) \]

\[ X = \sqrt{\alpha P} Q_1 + \sqrt{\alpha P} Q_2 \]
Scalar Gaussian BC with feedback

- Ozarow-Leung ’84: Schalkwijk-Kailath based scheme
Scalar Gaussian BC with feedback

 Oguz Ustun

- Ozarow-Leung ’84: Schalkwijk-Kailath based scheme
- Transmit at rates higher than $C_{no-FB}$
Scalar Gaussian BC with feedback

Ozarow-Leung ’84: Schalkwijk-Kailath based scheme
Transmit at rates higher than $C_{no-FB}$
Resolution

- Uncertainty at Rx 1: \( S_1 = \frac{U - E[U|Y_1X_0]}{c_1} \)
- Uncertainty at Rx 2: \( S_2 = \frac{V - E[V|Y_2X_0]}{c_2} \)
- Errors \( S_1, S_2 \) correlated & jointly Gaussian:
  \[
  S_1 = E[S_1 S_2] \cdot S_2 + Z
  \]
Resolution

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- Uncertainty at Rx 2: \( S_2 = \frac{V - E[V|Y_2X_0]}{c_2} \)
- Errors \( S_1, S_2 \) correlated & jointly Gaussian:
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  \]

Common Resolution Information

- Quantize \( S_2 \) to \( \hat{S}_2 \) with some distortion \( D \)
- Set \( C = \hat{S}_2, A = B = \phi \)
- \( P = P_1 + P_0 \): \( P_1 \) for fresh information, \( P_0 \) for resolution

\[
V = \sqrt{\alpha P_1} Q_1, \quad U = \sqrt{\alpha P_1} Q_2 + \frac{\alpha P_1}{\alpha P_1 + \sigma^2} V
\]

\[
X = \sqrt{\alpha P_1} Q_1 + \sqrt{\alpha P_1} Q_2 + \sqrt{\frac{P_0}{1 - D}} \hat{S}_2
\]
Achievable Rates with FB
Improvement

- We used only the common part of the errors - can do better!

- Sending correlated Gaussian sources over scalar Gaussian BC: [Soundarajan & Vishwanath ’08], [Bross, Lapidoth & Tinguely ’08] [Tian, Diggavi, Shamai ’10]: Complete distortion-rate region
  - Transmit quantized $S_2 +$ uncoded $S_1, S_2$

- We used estimation errors as the “sources” for resolution
  - Is there something better?
Block-Markov superposition coding scheme
- Feedback results in correlated information at receivers
- Resolve using joint source-channel coding

Start with Marton’s region with common part $W$
- Other degraded broadcast channels with feedback

Vector Gaussian broadcast channel

Can easily extend to noisy feedback