Bistable composite slit tubes
I: A beam model

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Abstract

Composite slit tubes with a circular cross-section show an interesting variety in their large-deformation behaviour, that depends on the layup of the surface that is used: tubes made from many antisymmetric laminae are bistable, and have a compact coiled configuration, tubes made from similar, but symmetric, laminae do not have a compact coiled state, and indeed may not be bistable, while tubes made from an isotropic sheet are not bistable. A simple model is presented here that is able to distinguish between these behaviours; it assumes that the cross-section remains circular, but allows twist, which is shown to be the key to making the distinction between the behaviours described.

Key words: Bistable, Composite

1 Introduction

The composite slit tubes described here are straight, thin-walled ribs with a cross-section that is a circular arc. When made from particular layups of composite, these structures have two stable states: an extended state where there is transverse curvature but no longitudinal curvature, and a coiled state

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Fig. 1. A bistable composite slit tube: (a) coiled; (b) extended. This paper will not consider the transition between (a) and (b) shown, but will assume that the entire tube is either coiled or extended. The global coordinate system $X, Y, Z$ used in the beam model is shown.

where there is longitudinal curvature and minimal transverse curvature; an example structure is shown in Figure 1.

The bistable behaviour shown in Figure 1 is not only a function of the geometry of the structure, but also of the bending and in-plane stiffness of the surface. Structures with similar geometry, but made from an isotropic metal sheet, are known as tape-springs (Seffen and Pellegrino, 1999), or STEM’s (Rimrott, 1966): STEM’s do not exhibit bistability, and require a spindle or casing to hold them when coiled. The bistable structure shown in Figure 1 is made from a laminate where the plies are antisymmetric with respect to the mid-surface. If the structure was made from a more conventional symmetric layup, the structure may not be bistable, and attempting to coil it results in the structure taking up a helical shape. This paper will describe a model that is able to reproduce these behaviours.

Previously Iqbal, Pellegrino & Daton-Lovett (2000) described a simple theoretical model for bistable composite slit tubes which calculates the total strain energy of the shell as a function of the transverse and longitudinal curvatures, and the angle subtended by the cross-section of the shell. This model provides an excellent insight into the behaviour of the bistable structures, and gives a good prediction of the behaviour of strips with a simple antisymmetric layup. However, because it restricts possible modes of deformation, it is not able to distinguish between the behaviour exhibited by symmetric laminates, antisymmetric laminates, and isotropic systems.

This paper, together with a companion paper, part II (Galletly and Guest, 2004), describe more comprehensive analytical models for composite slit tubes. One basic assumption, unchanged from Iqbal, Pellegrino & Daton-Lovett (2000),
Fig. 2. The initial geometric parameters for the bistable tube, $R$ and $l$, and the global coordinate system used for describing deformations and loads. The $X$-axis is in the longitudinal direction, and the $Y$-axis is in the transverse direction. The angle $\alpha = l/R$ will be used for presenting results in Section 3.

is that the structures are longitudinally uniform, and thus the models are making no attempt to model the transition from the rolled to the extended state, which takes place in a transition zone that can be seen in Figure 1. The models are instead trying to give information about the two possible stable shapes. Within the assumption of longitudinal uniformity, this paper allows the structure to take any shape longitudinally, whilst assuming that the cross-section remains as an arc of a circle, with a radius that is allowed to vary. Part II will, in addition, allow any cross-sectional shape.

One important assumption that is made both here and in Part II is that the tubes are initially unstressed, and hence we are not considering the behaviour of prestressed systems such as those recently described by Kebadze et al. (2004). However, it would be straightforward to incorporate the effect of prestress into the models described here.

The paper is structured as follows. Section 2 will describe the beam model, including its key innovation, that it allows the tubes to twist. Section 3 presents the results of applying this model to four sample tubes: one is made from an antisymmetric laminate; two are made from symmetric laminae; and one is made from an isotropic sheet. Section 4 presents a brief comparison with previous experiments and computations, and Section 5 concludes the paper.

2 Derivation of model

It is assumed that the tube is initially unstrained in its extended state, as shown in Figure 2. The tube geometry is defined by two length parameters, the width $l$, and the initial radius of curvature $R$; it is assumed to be infinitely long. To fully define the tube also requires the in-plane and bending stiffness properties of the shell to be defined; these are described in Section 2.3.
The beam model, described here, assumes that the beam deforms while remaining longitudinally uniform, with a cross-section that is an arc of a circle whose radius may change. Its configuration is then fully defined by four global strain parameters which will be described next, in Section 2.1. Section 2.2 describes the kinematic relationships between these global strain parameters, and local strains in the shell, and Section 2.3 describes the stress-strain relationship between these strains and the corresponding local stresses. Section 2.4 integrates these stresses to give global forces, work-conjugate to the global strain parameters: setting these forces to zero gives equilibrium configurations of the structure, and Section 2.5 describes the stability calculations for these configurations.

### 2.1 Global strain parameters

The assumption that the structure is uniform longitudinally allows six beam deformation modes: extension, $E_X$; twist, $\Phi_X$; bending about two axes, $K_Y$ and $K_Z$; and two shear modes, $\Gamma_{XY}$ and $\Gamma_{XZ}$. These modes are shown in Figure 3.

In fact, the beam model considers only three of the beam deformation modes shown in Figure 3. Two we can exclude by symmetry: the system has a two-fold rotational symmetry about the $Z$-axis, and $E_X$, $K_Y$, $\Phi_X$ and $\Gamma_{XY}$ are preserved by this symmetry operation, while $K_Z$ and $\Gamma_{XZ}$ are reversed. Thus, $K_Z$ and $\Gamma_{XZ}$ would only be of interest through some symmetry-breaking phenomenon that is not observed in experiments, and hence $K_Z$ and $\Gamma_{XZ}$ are assumed to be zero. We also exclude the shear modes, $\Gamma_{XY}$ and $\Gamma_{XZ}$ because we do not wish to explicitly constrain warping, and the deformations corresponding to $\Gamma_{XY}$ and $\Gamma_{XZ}$ are simply warping of the cross-section.

Thus the beam model considers three beam deformation modes, $E_X$, $K_Y$, $\Phi_X$, to which we add a further parameter to describe the transverse curvature of the section, $K_T$. Our model therefore describes the configuration of the slit tube by four parameters. In the initial state $E_X = K_Y = \Phi_X = 0$, and $K_T = 1/R$, and this is the first equilibrium point. These four global strain parameters fully define the shape of the structure; next we will consider the deformations of the surface, the local strains, that result from varying these four parameters.

### 2.2 Local strains

Changing the global strain parameters from the initial values causes deformation of the structure. This section considers the structure as a shell, and derives
expressions for the strains due to changes in the global strain parameters. The strains in the structure vary with position in the cross-section, described by the parameter $s$, shown in Figure 4; they are described in a local coordinate system $x(s)$, $y(s)$, $z(s)$. At the centre of the structure, these axes are aligned with the global system, but off-centre the $z$-axis remains perpendicular, and the $y$-axis parallel, to the surface.

To describe the deformation of a shell generally requires six generalised strain parameters. However, we only consider four of these, extension in the $x$-direction, $\epsilon_x$, curvature in the $x$ and $y$-directions, $\kappa_x$ and $\kappa_y$, and curvature of the surface, $\kappa_{xy}$. Extension transversely, $\epsilon_y$, and shear, $\gamma_{xy}$, are not defined by the global deformation modes, and we assume that these are not explicitly constrained. That shear deformation is allowed is consistent with our assumption in Section 2.1 that the surface is allowed to warp.
Fig. 4. A cross section of the beam.

Note our convention that global strains of the system, considered as a beam, are denoted using upper-case letters, while local strains of the surface, considered as a shell, are denoted using lower-case letters. It is also worth noting here that treating the structure as both a beam and a shell leads to potential confusion because of the differing standard notations for curvature. As a beam, where the $X$-axis defines the longitudinal direction, $K_Y$ describes the rotation of the cross-section around the $Y$-axis as one moves along the beam. As a shell, where the $z$-axis describes a normal to the surface, $\kappa_x$ describes a similar rotation of the surface around the $y$-axis as one moves in the $x$-direction. Thus along the centre-line of the beam, where $x,y,z$ and $X,Y,Z$ coincide, $\kappa_x = K_Y$.

2.2.1 Strains due to twisting

The strains due to twisting are the most complex part of this exposition, and hence are detailed separately.

We make the assumption that due to twisting, each section rotates about the centre-line of the structure; choice of rotation axis does not affect the resultant local twist of the surface (Timoshenko and Gere, 1961, Chapter 5), but does affect the longitudinal extension. An alternative, more difficult, assumption, would be that each section rotates about its shear centre. However, for small $K_T$ the expressions reduce to the same values, which will be described in Section 2.2.5.

We note here the convention that we will use for twist of the surface, $\kappa_{xy}$. Literature on shell and plate theory, and on composite theory, use different conventions (similar to the distinction between engineering and mathematical shear strain). We shall use the composite theory definition, $\kappa_{xy} = -2\partial^2 w/\partial x \partial y$, where $w$ is the displacement of the surface in the out-of-plane, $z$, direction, as defined by Halpin (1984) (the literature on shell and plate theory would define $\kappa_{xy}$ to be half this value).

Consider the small section of surface shown in Figure 5. The rotation of a cross-
section at a distance $x$ from a reference section is given by $\theta = \Phi_X x$, and the displacement in the $z$-direction of a point at $(x, y)$ is given by $w = \theta y = \Phi_X xy$. Thus

$$\Phi_X = \frac{\partial^2 w}{\partial x \partial y}$$

(1)

and $\kappa_{xy}$ is given by

$$\kappa_{xy} = -2 \frac{\partial^2 w}{\partial x \partial y} = -2 \Phi_X$$

(2)

The global twist mode, $\Phi_X$, will cause longitudinal fibres to take up a helical shape, and hence also cause some extension. A longitudinal fibre at a distance $c$ from the axis of rotation will end up at an orientation $c \Phi_X$ to the longitudinal axis. A short length of fibre is shown in Figure 6, and will (neglecting higher-order terms), undergo a strain $\epsilon$,

$$\epsilon = \frac{\Phi_X^2 c^2}{2}$$

(3)

The length $c$, as shown in Figure 7 is given by

$$c = \frac{2}{K_T} \sin \left( \frac{sK_T}{2} \right)$$

(4)

and hence

$$\epsilon = \frac{2 \Phi_X^2}{K_T} \sin^2 \left( \frac{sK_T}{2} \right)$$

(5)
Fig. 7. A cross-section of the beam, showing the distance from the central fibre.

2.2.2 Longitudinal strain

Beam extension, beam bending, and beam twisting (Equation 5) will cause longitudinal strain:

\[ \epsilon_x(s) = E_X - d(s) K_Y + \frac{2\Phi_X^2}{K_T^2} \sin^2 \left( \frac{sK_T}{2} \right) \]  \hspace{1cm} (6)

where \( d(s) \) is the distance from the neutral axis for bending about the \( Y \)-axis, as shown in Figure 4.

Geometry shows that \( d(s) \) is given by

\[ d(s) = \frac{1}{K_T} \left[ \frac{2}{lK_T} \sin \left( \frac{lK_T}{2} \right) - \cos sK_T \right] \]  \hspace{1cm} (7)

and hence

\[ \epsilon_x(s) = E_X - \frac{K_Y}{K_T} \left[ \frac{2}{lK_T} \sin \left( \frac{lK_T}{2} \right) - \cos sK_T \right] + \frac{2\Phi_X^2}{K_T^2} \sin^2 \left( \frac{sK_T}{2} \right) \]  \hspace{1cm} (8)

2.2.3 Longitudinal curvature

A component of beam curvature will cause local longitudinal curvature,

\[ \kappa_x(s) = K_Y \cos sK_T \]  \hspace{1cm} (9)

The angle \( sK_T \) is shown in Figure 4.

2.2.4 Transverse curvature

\( \kappa_y \) is equal to the change in the transverse curvature, \( K_T \), from its initial value \( 1/R \).

\[ \kappa_y = K_T - \frac{1}{R} \]  \hspace{1cm} (10)
2.2.5 Kinematic relationships for small cross-sectional curvature

Experience with using the model described here shows that the interesting behaviour occurs when the cross-sectional curvature is small, which allows the expressions derived to be simplified. Here we present simplified equations for the local strains that will be used in later calculations, found from Equations 8, 9, 10 and 2 by assuming that $K_T$ is small, and neglecting $K_T^2$, and higher powers.

\[
\epsilon_x = E_X + \frac{K_Y K_T}{2} \left[ \frac{l_x^2}{12} - s^2 \right] + \frac{\Phi_X s^2}{2} \quad (11)
\]

\[
\kappa_x = K_Y \quad (12)
\]

\[
\kappa_y = K_T - \frac{1}{R} \quad (13)
\]

\[
\kappa_{xy} = -2\Phi_X \quad (14)
\]

2.3 Stress-strain relationship

Although the deflections in the structure are large, we assume that the strains remain small, and the general, linear stress-strain relationship for a composite material is then given by:

\[
\begin{pmatrix}
N_x \\
N_y \\
N_{xy} \\
M_x \\
M_y \\
M_{xy}
\end{pmatrix}
= 
\begin{bmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\
- & - & - & - & - & - \\
B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\
B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\
B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66}
\end{bmatrix}
\begin{pmatrix}
\epsilon_x \\
\epsilon_y \\
\gamma_{xy} \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{pmatrix}
\quad (15)
\]

where $\epsilon_x$, $\epsilon_y$, $\gamma_{xy}$, $\kappa_x$, $\kappa_y$ and $\kappa_{xy}$ are the generalised strains, already introduced, and $N_x$, $N_y$, $N_{xy}$, $M_x$, $M_y$ and $M_{xy}$ are the corresponding stress resultants. The $A_{ij}$ are the laminate extensional stiffnesses, relating extensional and shear forces to extensional and shear strains. The $B_{ij}$ are the laminate coupling stiffnesses, relating extensional and shear forces to bending and twisting curvatures, and bending and twisting moments to extensional and shear strains. The $D_{ij}$ are the laminate bending stiffnesses, relating bending and twisting moments to bending and twisting curvatures. The theory explaining how these laminate stiffnesses are calculated can be found in any good primer on composite materials, for example Hull (1981) or Halpin (1984).
In this model, no forces are applied at the edges of the tube. Hence \(N_y = N_{xy} = 0\) here, and we assume that this is true throughout the tube; correspondingly, we take \(\epsilon_y\) and \(\gamma_{xy}\) to be unconstrained. We can rewrite Equation 15 as

\[
\begin{bmatrix}
N_x \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & B_{11} & B_{12} & B_{16} & A_{12} & A_{16} \\
B_{11} & D_{11} & D_{12} & D_{16} & B_{12} & B_{16} \\
B_{12} & D_{12} & D_{22} & D_{26} & B_{22} & B_{26} \\
B_{16} & D_{16} & D_{26} & D_{66} & B_{26} & B_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_x \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\]

(16)

Thus we find that \(\epsilon_y\) and \(\gamma_{xy}\) are given by

\[
\begin{bmatrix}
\epsilon_y \\
\gamma_{xy}
\end{bmatrix} =
\begin{bmatrix}
A_{22} & A_{26} \\
A_{26} & A_{66}
\end{bmatrix}^{-1}
\begin{bmatrix}
A_{12} & B_{12} & B_{22} & B_{26} \\
A_{16} & B_{16} & B_{26} & B_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_x \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\]

(17)

and the remaining stress resultants are given by

\[
\begin{bmatrix}
N_x \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix} =
\begin{bmatrix}
A_{11}^* & B_{11}^* & B_{12}^* & B_{16}^* \\
B_{11}^* & D_{11}^* & D_{12}^* & D_{16}^* \\
B_{12}^* & D_{12}^* & D_{22}^* & D_{26}^* \\
B_{16}^* & D_{16}^* & D_{26}^* & D_{66}^*
\end{bmatrix}
\begin{bmatrix}
\epsilon_x \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\]

(18)

which we write as \(N = [K^*]_\xi\), where the reduced stiffness matrix \([K^*]\) is given by:

\[
[K^*] =
\begin{bmatrix}
A_{11} & B_{11} & B_{12} & B_{16} \\
B_{11} & D_{11} & D_{12} & D_{16} \\
B_{12} & D_{12} & D_{22} & D_{26} \\
B_{16} & D_{16} & D_{26} & D_{66}
\end{bmatrix}
- \begin{bmatrix}
A_{11} & A_{16} \\
B_{12} & B_{16} \\
B_{22} & B_{26} \\
B_{26} & B_{66}
\end{bmatrix}^{-1}
\begin{bmatrix}
A_{12} & B_{12} & B_{22} & B_{26} \\
A_{16} & B_{16} & B_{26} & B_{66}
\end{bmatrix}
\]

(19)

For any antisymmetric layup, symmetry arguments show that \(B_{11}^* = B_{12}^* = D_{16}^* = D_{26}^* = 0\). Similarly, for a symmetric layup, \(B_{11}^* = B_{12}^* = B_{16}^* = 0\). 

10
The systems in this paper will all either be antisymmetric, or symmetric, and
\( B_{11}^* = B_{12}^* = 0 \) will be used to simplify the algebra in the next section.

2.4 Derivation of global forces

Corresponding in a work sense to each of the global strain parameters, \( E_X, K_Y, \Phi_X \), \( K_T \) will be a global force: a tension \( P \), a longitudinal moment \( M_L \), a torque \( T \) and a transverse moment/unit length, \( M_T \). To find an equilibrium position, we are looking for a position where each of these equals zero.

2.4.1 Global Tension

The global tension is given by

\[
P = \int_{-\frac{l}{2}}^{\frac{l}{2}} N_x \, ds.
\]

For the symmetric and antisymmetric sections that we are considering, from Equation 18 with \( B_{11}^* = B_{12}^* = 0 \),

\[
N_x = A_{11}^* \epsilon_x + B_{16}^* \kappa_{xy}
\]

and substituting from the kinematic relationships Equations 11 and 14,

\[
N_x = A_{11}^* \left( E_X + \frac{K_Y K_T}{2} \left[ \frac{l^2}{12} - s^2 \right] + \frac{\Phi_X^2 s^2}{2} \right) - 2B_{16}^* \Phi_X
\]

Thus, completing the integration

\[
P = A_{11}^* \left( E_X l + \frac{\Phi_X^2 l^3}{24} \right) - 2B_{16}^* \Phi_X l
\]

As will be seen later, the global strain \( E_X \) only occurs in the expression for this global force, and not any other. Thus, it is straightforward to ensure that \( P = 0 \) by finding an expression for \( E_X \) and substituting this algebraically into the other kinematic relationships.

Setting \( P = 0 \) gives

\[
E_X = \frac{2B_{16}^* \Phi_X}{A_{11}^*} - \frac{\Phi_X^2 l^2}{24}
\]

and substituting into expressions for the strains, Equations 11, 12, 13 and 14 gives:
\[ \epsilon_x = \frac{K_Y K_T}{2} \left( \frac{l^2}{12} - s^2 \right) + \frac{\Phi_X^2}{2} \left( s^2 - \frac{l^2}{12} \right) + 2B^*_1 \Phi_X \]  
\[ \kappa_x = K_Y \]  
\[ \kappa_y = K_T - \frac{1}{R} \]  
\[ \kappa_{xy} = -2\Phi_X \]  
(25)  
(26)  
(27)  
(28)

These expressions, functions of only \( K_Y, K_T \) and \( \Phi_X \), will be used for calculating the other global forces.

### 2.4.2 Global Torque

It was clear how to find the global tension in the beam, but the relationships are not so simple for the three remaining global forces. We shall calculate these using Virtual Work.

To calculate the torque, we consider that if a small change in the twist \( \delta \Phi^*_X \) occurs, the internal work done must be equal to the external work. The external work per unit length is given by \( T \delta \Phi^*_X \). To calculate the internal work, we must first calculate the changes in the strains which are compatible with \( \delta \Phi^*_X \).

\[ \delta \epsilon = \frac{\partial \epsilon}{\partial \Phi} \delta \Phi^*_X \]  
(29)

where \( \epsilon \) is the initial strain vector, and \( \delta \epsilon \) is the vector of strain increments. The internal work done per unit length is thus given by

\[ W = \int_{-\frac{l}{2}}^{\frac{l}{2}} \delta \epsilon^T \cdot N \, ds = \int_{-\frac{l}{2}}^{\frac{l}{2}} \delta \Phi^*_X \cdot \frac{\partial \epsilon}{\partial \Phi} \cdot N \, ds \]  
(30)

where \( N \) is the vector of internal forces. Equating the internal and external work, and cancelling both sides by \( \delta \Phi^*_X \) we find an expression for the torque:

\[ T = \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{\partial \epsilon}{\partial \Phi} \cdot N \, ds \]  
(31)

Expanding this out gives

\[ T = \int_{-\frac{l}{2}}^{\frac{l}{2}} \left( N_x \frac{\partial \epsilon_x}{\partial \Phi} + M_x \frac{\partial \kappa_x}{\partial \Phi} + M_y \frac{\partial \kappa_y}{\partial \Phi} + M_{xy} \frac{\partial \kappa_{xy}}{\partial \Phi} \right) \, ds \]  
(32)

As \( \frac{\partial \kappa_x}{\partial \Phi} = \frac{\partial \kappa_y}{\partial \Phi} = 0 \) this simplifies to:

\[ T = \int_{-\frac{l}{2}}^{\frac{l}{2}} \left( N_x \frac{\partial \epsilon_x}{\partial \Phi} + M_{xy} \frac{\partial \kappa_{xy}}{\partial \Phi} \right) \, ds \]  
(33)
After substituting for these terms, and integrating with respect to $s$, we obtain the following:

$$T = A_{11}^* \left[ \left( \Phi_X - K_T \Phi_X \right) \frac{K_T l^5}{360} \right] - 2D_{16}^* l K_T Y - 2D_{26}^* l \left( K_T - \frac{1}{R} \right) + 4 \left( D_{66}^* - \frac{B_{16}^2}{A_{11}^*} \right) l \Phi_X \tag{34}$$

### 2.4.3 Global longitudinal moment

A virtual work approach shows that the longitudinal moment, $M_l$ is given by:

$$M_l = \int_{-\frac{l}{2}}^{\frac{l}{2}} \left( N_x \frac{\partial \varepsilon_x}{\partial K_Y} + M_x \frac{\partial \kappa_x}{\partial K_Y} + M_y \frac{\partial \kappa_y}{\partial K_Y} + M_{xy} \frac{\partial \kappa_{xy}}{\partial K_Y} \right) ds \tag{35}$$

As $\frac{\partial \kappa_y}{\partial K_Y} = \frac{\partial \kappa_{xy}}{\partial K_Y} = 0$ this simplifies to

$$M_l = \int_{-\frac{l}{2}}^{\frac{l}{2}} \left( N_x \frac{\partial \varepsilon_x}{\partial K_Y} + M_x \frac{\partial \kappa_x}{\partial K_Y} \right) ds \tag{36}$$

After substituting for these terms, and integrating with respect to $s$, we obtain the following:

$$M_l = A_{11}^* \left[ \left( K_Y K_T - \Phi_X \right) \frac{K_T l^5}{720} \right] + D_{11}^* l K_Y + D_{12}^* l \left( K_T - \frac{1}{R} \right) - 2D_{16}^* l \Phi_X \tag{37}$$

### 2.4.4 Global transverse moment

A virtual work approach shows that the transverse moment, $M_T$ is given by:

$$M_T = \int_{-\frac{l}{2}}^{\frac{l}{2}} \left( N_x \frac{\partial \varepsilon_x}{\partial K_T} + M_x \frac{\partial \kappa_x}{\partial K_T} + M_y \frac{\partial \kappa_y}{\partial K_T} + M_{xy} \frac{\partial \kappa_{xy}}{\partial K_T} \right) ds \tag{38}$$

As $\frac{\partial \kappa_x}{\partial K_T} = \frac{\partial \kappa_{xy}}{\partial K_T} = 0$ this simplifies to

$$M_T = \int_{-\frac{l}{2}}^{\frac{l}{2}} \left( N_x \frac{\partial \varepsilon_x}{\partial K_T} + M_y \frac{\partial \kappa_y}{\partial K_T} \right) ds \tag{39}$$

After substituting for these terms, and integrating with respect to $s$, we obtain the following:
2.5 Stability criteria

Finding an equilibrium point requires that \( T = M_l = M_T = 0 \), but it is also necessary to know about the stability of this equilibrium. This can be determined by examining the local tangent-stiffness matrix, \([H]\), given in Equation 41. If \([H]\) is positive-definite, the equilibrium will be stable; the test for positive-definiteness used in this paper is that the smallest eigenvalue of \([H]\), the stiffness of the softest mode of deformation, must be positive.

\[
[H] = \begin{bmatrix}
\frac{\partial T}{\partial \Phi_X} & \frac{\partial M_l}{\partial \Phi_X} & \frac{\partial M_T}{\partial \Phi_X} \\
\frac{\partial T}{\partial K_Y} & \frac{\partial M_l}{\partial K_Y} & \frac{\partial M_T}{\partial K_Y} \\
\frac{\partial T}{\partial K_T} & \frac{\partial M_l}{\partial K_T} & \frac{\partial M_T}{\partial K_T}
\end{bmatrix}
\]  

(41)

where,

\[
\frac{\partial T}{\partial \Phi_X} = A_{11}^* \left[ \left( 3\Phi_X^2 - K_YK_T \right) \frac{l^5}{360} \right] + 4 \left( D_{66}^* - \frac{B_{16}^*}{A_{11}^*} \right) l
\]

\[
\frac{\partial M_l}{\partial \Phi_X} = -A_{11}^* \Phi_X \frac{K_T l^5}{360} - 2D_{16}^* l
\]

\[
\frac{\partial M_T}{\partial \Phi_X} = -A_{11}^* \Phi_X \frac{K_Y l^5}{360} - 2D_{26}^* l
\]

\[
\frac{\partial M_T}{\partial K_Y} = A_{11}^* \frac{K_Y^2 l^5}{720} + D_{22}^* l
\]

\[
\frac{\partial M_l}{\partial K_Y} = A_{11}^* \left[ \left( 2K_YK_T - \Phi_X^2 \right) \frac{l^5}{720} \right] + D_{12}^* l
\]

\[
\frac{\partial M_T}{\partial K_T} = A_{11}^* \frac{K_T^2 l^5}{720} + D_{11}^* l
\]

(42)

The matrix \([H]\) is, of course, symmetric: \( \partial T/\partial K_Y = \partial M_l/\partial \Phi_X, \partial T/\partial K_T = \partial M_T/\partial \Phi_X \) and \( \partial M_T/\partial K_Y = \partial M_l/\partial K_T \).
3 Results

Results are presented for four sample cases which are chosen to show the range of possible behaviours for these systems, and to allow some comparison with previously published experimental results, in Section 4. For each case we show:

- simplified expressions for the torque and moments, taking into account the appropriate zero terms in the reduced stiffness matrix;
- the reduced stiffness matrix;
- the position of equilibrium points other than the original, for varying $\alpha$ (shown in Figure 2), and whether these equilibria are stable.

The results are presented for three tubes made from different laminae, and one tube made from an isotropic sheet. The laminate results presented are for 5-layer laminates; each ply is 0.21 mm thick, with a polypropylene matrix containing 30% volume fraction of glass fibres. One antisymmetric laminate is chosen that has been studied previously, both experimentally and computationally; the layers are laid up with the glass at $[+45^\circ, -45^\circ, 0^\circ, +45^\circ, -45^\circ]$ to the longitudinal direction. A similar symmetric laminate is also presented with the glass at $[+45^\circ, -45^\circ, 0^\circ, -45^\circ, +45^\circ]$ to the longitudinal direction. This proves not to have a second stable state for practical systems, and hence a second symmetric laminate is presented that does have a second stable state, where the layup is $[+40^\circ, -40^\circ, 0^\circ, -40^\circ, +40^\circ]$. The isotropic results are presented for a 0.125 mm thick steel sheet. Further details of the materials will be found in Galletly (2001).

3.1 Antisymmetric layup

For an antisymmetric layup, where $D_{16}^* = D_{26}^* = 0$, the expressions for the torque, and for the longitudinal and transverse moments, reduce to:

$$T = A_{11}^* \left[ (\Phi_X^2 - K_Y K_T) \frac{\Phi_X l^5}{360} \right] + 4 \left( D_{66}^* - \frac{B_{16}^2}{A_{11}^*} \right) l \Phi_X \quad (43)$$

$$M_l = A_{11}^* \left[ (K_Y K_T - \Phi_X^2) \frac{K_Y l^5}{720} \right] + D_{11}^* l K_Y + D_{12}^* l \left( K_T - \frac{1}{R} \right) \quad (44)$$

$$M_T = A_{11}^* \left[ (K_Y K_T - \Phi_X^2) \frac{K_T l^5}{720} \right] + D_{12}^* l K_Y + D_{22}^* l \left( K_T - \frac{1}{R} \right) \quad (45)$$
The reduced stiffness matrix for the $[+45^\circ, -45^\circ, 0^\circ, +45^\circ, -45^\circ]$ layup is:

$$
\begin{pmatrix}
N_x \\
M_x \\
M_y \\
M_{xy}
\end{pmatrix} =
\begin{pmatrix}
8.757 & 0 & 0 & 0.1361 \\
0 & 0.819 & 0.616 & 0 \\
0 & 0.616 & 0.799 & 0 \\
0.1361 & 0 & 0 & 0.643
\end{pmatrix}
\begin{pmatrix}
\epsilon_x \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{pmatrix}
$$

The units for this matrix are: GPa mm for the top left-hand term ($A_{11}^*$), GPa mm$^2$ for the top right-hand 1 × 3 and (identically) for the bottom left-hand 3 × 1 matrix ($B_{11}^*, B_{12}^*, B_{16}^*$) and GPa mm$^3$ for the bottom right-hand 3 × 3 matrix ($D_{11}$).

Solving Equations 43–45 numerically allows equilibrium points ($T = M_l = M_T = 0$) in addition to the original state to be found for $\alpha$ greater than a critical value, which for this example is $\alpha \geq 56^\circ$. Figure 8 shows how the values of the strains at the equilibrium point vary with the initial cross-sectional angle $\alpha$. A number of different solutions are possible, but only one is stable. The stable mode has $\Phi_X = 0$, and thus for this case a simpler model that neglects twist is adequate.

For large values of $\alpha$, the stable solution tends towards $K_Y = 0.752/R$, $K_T = 0$, with $\Phi_X=0$.

### 3.2 Symmetric layup

For a symmetric layup, where $B_{16}^* = 0$, the expressions for the torque, and for the longitudinal and transverse moments, reduce to:

$$T = A_{11}^* \left[ \left( \Phi_X^2 - K_Y K_T \right) \frac{K_T l^5}{360} \right] - 2D_{16}^* l K_Y - 2D_{26}^* l \left( K_T - \frac{1}{R} \right) + 4D_{66}^* l \Phi_X$$ (47)

$$M_l = A_{11}^* \left[ \left( K_Y K_T - \Phi_X^2 \right) \frac{K_T l^5}{720} \right] + D_{11}^* l K_Y + D_{12}^* l \left( K_T - \frac{1}{R} \right) - 2D_{16}^* l \Phi_X$$ (48)

$$M_T = A_{11}^* \left[ \left( K_Y K_T - \Phi_X^2 \right) \frac{K_Y l^5}{720} \right] + D_{12}^* l K_Y + D_{22}^* l \left( K_T - \frac{1}{R} \right) - 2D_{26}^* l \Phi_X$$ (49)
Fig. 8. Equilibrium points in addition to the initial state for the 45° antisymmetric layup. There is only one stable state, shown by the solid line, and this exists for \( \alpha \geq 56° \). This solution has a corresponding unstable branch shown by the dashed line. At \( \alpha = 69° \) a bifurcation leads to two additional twisted unstable solutions (one with positive twist, one with negative twist), shown by dotted lines.

### 3.2.1 45° layup

The reduced stiffness matrix for the \([+45°, -45°, 0°, -45°, +45°]\) layup is:

\[
\begin{bmatrix}
N_x \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix}
= \begin{bmatrix}
8.757 & 0 & 0 & 0 \\
0 & 0.868 & 0.665 & 0.345 \\
0 & 0.665 & 0.848 & 0.345 \\
0 & 0.345 & 0.345 & 0.681
\end{bmatrix}
\begin{bmatrix}
\epsilon_x \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\]  

(50)

The units for this matrix are identical to those for the matrix in Equation 46.

Solving Equations 47–49 numerically shows that there are the only stable solution in addition to the original state exists only for very large \( \alpha \). In the range of practical interest, e.g. \( \alpha \leq 360° \), there are no additional stable solutions; unstable twisted solutions appear for \( \alpha \geq 112° \). Figure 9 shows how the values of the strains for the unstable equilibria vary with the initial cross-sectional angle \( \alpha \).
3.2.2 40° layup

Although the additional stable solution for the symmetric 45° layup exists only for very large $\alpha$, it would be wrong to conclude that this is always the case. Here we present a 40° layup that does have a second stable equilibrium solution for $\alpha$ in the range of practical interest.

The reduced stiffness matrix for the $[+40^\circ, -40^\circ, 0^\circ, -40^\circ, +40^\circ]$ layup is:

$$
\begin{bmatrix}
    N_x \\
    M_x \\
    M_y \\
    M_{xy}
\end{bmatrix} =
\begin{bmatrix}
    9.844 & 0 & 0 & 0 \\
    0 & 1.092 & 0.647 & 0.398 \\
    0 & 0.647 & 0.660 & 0.281 \\
    0 & 0.398 & 0.281 & 0.663
\end{bmatrix}
\begin{bmatrix}
    \epsilon_x \\
    \kappa_x \\
    \kappa_y \\
    \kappa_{xy}
\end{bmatrix}
$$

The units for this matrix are identical to those for the matrix in Equation 46.

Solving Equations 47–49 numerically shows that, in addition to the original state, there is a stable twisted solution, that appears for $\alpha \geq 89^\circ$, as well as the unstable solutions that are little altered from those presented in Figure 9. Figure 10 shows how the values of the strains at the equilibrium point vary...
Fig. 10. Equilibrium points in addition to the initial state for the $40^\circ$ symmetric layup. There is only one stable state, shown by the solid line, and this exists for $\alpha \geq 89^\circ$. This solution has a corresponding unstable branch shown by the dashed line. Additional unstable solutions are shown by dotted lines.

with the initial cross-sectional angle $\alpha$. Note that here, the stable state found is twisted, with $\Phi_X \neq 0$.

For large values of $\alpha$, the stable solution tends towards $K_Y = 0.503/R$, $K_T = 0.018/R$, $\Phi_X = -0.095/R$.

3.3 Isotropic case

For the isotropic case, the expressions for the torque, and for the longitudinal and transverse moments, reduce to:

$$T = Et \left[ \left( \Phi_X^2 - K_Y K_T \right) \frac{\Phi_X l^5}{360} \right] + 2Dl (1 - \nu) \Phi_X \quad (52)$$

$$M_l = Et \left[ \left( K_Y K_T - \Phi_X^2 \right) \frac{K_T l^5}{720} \right] + DK_Y l + \nu Dl \left( K_T - \frac{1}{R} \right) \quad (53)$$

$$M_T = Et \left[ \left( K_Y K_T - \Phi_X^2 \right) \frac{K_Y l^5}{720} \right] + \nu DK_Y l + Dl \left( K_T - \frac{1}{R} \right) \quad (54)$$
The reduced stiffness matrix of an isotropic sheet (Calladine, 1983, adjusted for the definition of $\kappa_{xy}$) is:

\[
\begin{pmatrix}
N_x \\
M_x \\
M_y \\
M_{xy}
\end{pmatrix} =
\begin{pmatrix}
Et & 0 & 0 & 0 \\
0 & D & \nu D & 0 \\
0 & \nu D & D & 0 \\
0 & 0 & 0 & \frac{D(1-\nu)}{2}
\end{pmatrix}
\begin{pmatrix}
\epsilon_x \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{pmatrix}
\]  

(55)

Where $E$ is the Young’s modulus of the material and $\nu$ the Poisson’s ratio, $t$ is the thickness of the material, and $D = Et^3/12(1-\nu^2)$. For a 0.125 mm thick steel sheet this becomes

\[
\begin{pmatrix}
N_x \\
M_x \\
M_y \\
M_{xy}
\end{pmatrix} =
\begin{pmatrix}
25.875 & 0 & 0 & 0 \\
0 & 0.0370 & 0.0111 & 0 \\
0 & 0.0111 & 0.0370 & 0 \\
0 & 0 & 0 & 0.0130
\end{pmatrix}
\begin{pmatrix}
\epsilon_x \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{pmatrix}
\]  

(56)

The units for this matrix are identical to those for the matrix in Equation 46.

Solving Equations (52–54) numerically shows that there are no stable solutions in this case. Figure 11 shows how the values of the strains at the equilibrium point vary with the initial cross-sectional angle $\alpha$. Unstable solutions appear for $\alpha \geq 62^\circ$; for these solutions $\Phi_X$ is always zero. Because of this, a model that neglects twisting would find this equilibrium point; however, it would not correctly predict the stability of the equilibrium, as the unstable mode of deformation involves twisting. These solutions correspond to the stable/unstable pair of solutions for the 45° antisymmetric layup, but in this case both branches are unstable.

4 Comparison with experimental results

For the 45° antisymmetric layup, the values given by the beam model for the longitudinal radius of curvature in the coiled state, $1/K_y$, can be compared with the results of experiments and finite element analyses reported by Iqbal, Pellegrino & Daton-Lovett (2000). The finite element results presented were obtained by re-running Iqbal’s analyses using the material properties given in this paper, which differ slightly from those used by Iqbal. The method is described in detail by Iqbal (2001), and a summary of the method will be given in Part II, where more detailed cross-sectional shapes are considered. The comparison is given in Table 1.
There are only unstable equilibria, which exist for $\alpha \geq 62^\circ$, shown by dotted lines.

Table 1
The coiled longitudinal radius of curvature, $1/K_y$, found using the beam model, compared with experiments and finite element calculations for the antisymmetric $45^\circ$ layup with $R = 29$ mm. The experimental results are taken from Iqbal, Pellegrino & Daton-Lovett (2000), and the finite element results are found using the method described in Iqbal (2001).

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>experiment (mm)</th>
<th>beam model (mm)</th>
<th>finite element (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>280°</td>
<td>30</td>
<td>38.5</td>
<td>39.6</td>
</tr>
<tr>
<td>200°</td>
<td>30</td>
<td>38.5</td>
<td>40.5</td>
</tr>
<tr>
<td>120°</td>
<td>32</td>
<td>38.7</td>
<td>43.0</td>
</tr>
<tr>
<td>90°</td>
<td>33</td>
<td>39.2</td>
<td>45.7</td>
</tr>
</tbody>
</table>

It can be seen from Table 1 that the results from the beam model correlate fairly well with the finite element results (particularly for large cross-sectional values of $\alpha$), but less well with the experimental results. A possible reason for the disparity between the calculated and experimental results is that, at high strains, the matrix material is not linear elastic, but instead exhibits viscoelastic behaviour. The beam model presented in this paper, and finite element calculations, have all assumed linear elastic behaviour. If the disparity...
was due to viscoelastic effects, the beam model should correlate well with the instantaneous experimental radius at the moment of snap-through. The experimental measurements reported by Iqbal, Pellegrino & Daton-Lovett (2000) were taken 24 hours after snap-through. A brief experiment to try to capture an instantaneous radius was conducted; this measured a value of 37 mm, which accords much better with the calculated results.

The paucity of experimental data, and the poor agreement between the experimental and linear-elastic analytical and computational models, shows the need for further experimental and computational studies; work on this continues.

5 Conclusions

This paper has shown that modelling twist is essential to distinguishing between slit tubes that are bistable and those that are not. For many antisymmetric layups, twist is unimportant, and a model that neglects twist is adequate, but it will not be able to determine if the model is unstable in a twisting mode. For symmetric layups the second equilibrium point, if it exists, is twisted. For isotropic systems, a second equilibrium point exists that is not twisted, but it is unstable.

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References


