The stiffness of prestressed frameworks: a unifying approach

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Abstract

A simple derivation of the tangent stiffness matrix for a prestressed pin-jointed structure is given, and is used to compare the diverse formulations that can be found in the literature for finding the structural response of prestressed structures.

1 Introduction

This paper gives a simple derivation of the tangent stiffness matrix for a prestressed pin-jointed structure: the stiffness is found by differentiating equilibrium expressions at nodes of the structure with respect to the position of the nodes. It uses this derivation to compare the diverse formulations that are applied to understanding the mechanics of prestressed structures by different academic communities.

Two basic approaches to understanding the mechanics of pin-jointed structures are common. In the computational mechanics approach, the results of computations are used to gain insight into structural response. In this context, it is sensible to use an exact tangent stiffness matrix, as described by e.g. Argyris and Scharpf (1972). Another approach is to gain understanding through the basic formulation of the problem: an exact formulation is less important, and it may prove sensible to use a simplified set of equations. This paper shows the links between various such formulations by describing the exact tangent stiffness matrix using *equilibrium* and *stress* matrices, each of which has been used individually to gain understanding of structural response in different circumstances.

Using the equilibrium matrix to understand structural response is described in e.g. Pellegrino and Calladine (1986), or Pellegrino (1993). A basic assumption is that the key structural action comes through the deformation of members — a common assumption in structural engineering. Study of the equilibrium matrix (or equivalently its transpose, the *compatibility* matrix) enables small movements of the structure to be decomposed into movements that cause deformation of members, and *mechanisms* that to a first order approximation cause no deformation of members. It is also possible to find various *states of self-stress*, where the structure is stressed even under zero external load. The fact that a structure has a mechanism (by the definition given here) does not imply that this motion has no stiffness as long as the structure is stressed, and Pellegrino (1990) and Calladine and Pellegrino (1991) further describe a method where this stiffness may be found using *product forces*. This paper will show that in fact this extension corresponds to a reduced form of the stress matrix, described next.

The *stress matrix* is widely used in the mathematical rigidity theory literature, see e.g. Connelly and Terrell (1995) or Connelly and Back (1998). Here, the basic structural action is assumed to come about through the reorientation of stressed bars. The aim of this work is not conventional modelling of structures, but answering questions such as when a particular set of links implies a unique configuration of nodes. Of particular relevance here is that the stress matrix is used to understand whether unconventional structures such as tensegrities are 'prestress stable' (Connelly and Whiteley, 1996).

This paper will show that to find structural stiffness, the equilibrium matrix, and the stress matrix, are usefully complementary. When combined with the definition in this paper of a 'modified axial stiffness' for a prestressed bar, the equilibrium matrix and the stress matrix together can be used to give the correct tangent stiffness matrix, without sacrificing the useful insight that the simplified methods give.

The paper is structured as follows. This introduction will conclude by introducing an example structure. Section 2 will describe the formulation itself, and this will be compared with earlier work in Section 3. The example structure will be analysed in Section 4, and Section 5 will conclude the paper.

1.1 Introduction to the example structure

The structure shown in Figure 1 will be used as an example. Considered in 2D, with out-of-plane motion restricted, the structure has no mechanisms: conventional structural action renders it stiff. Considered in 3D, however, there is a mechanism in which the completely unrestrained joint moves out of plane. The structure can sustain a state of self-stress, with the two cross-bars in compression and the outer bars in tension, or vice versa, and the tangent stiffness matrix will be used to clarify if the state of self-stress will stiffen the out-of-plane mechanism.

2 Tangent stiffness formulation

This section introduces a new derivation for the tangent stiffness, found by initially writing down the equations of equilibrium for the external forces at each of the nodes of the structure, and then differentiating these forces with respect to movement of the nodes. For simplicity, the tangent stiffness will first be found for a single bar, before a general

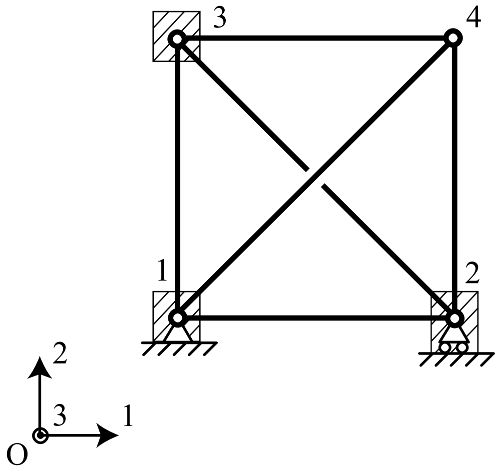

Figure 1: A simple example structure, analysed in Section 4. It consists of four joints, numbered 1 to 4, all of which lie in the 1-2 plane. Joint 1 is fully restrained, joint 2 is allowed to move only in the 1-direction, joint 3 is retrained to lie in the 1-2 plane, and joint 4 is completely free. The joints are connected by six bars; the two crossing bars are not connected.

pin-jointed structure is considered. The derivation will equally apply in two or three dimensions.

2.1 A single bar

Figure 2 shows a single bar floating in space. Forces $\mathbf{f}_1$ and $\mathbf{f}_2$ are in equilibrium with an internal tension in the bar t, where $\mathbf{f}_1$ and $\mathbf{f}_2$ are two- or three-dimensional vectors as appropriate, with components f_{1i} and f_{2i} respectively. The nodes of the bar have position vectors, $\mathbf{x}_1$ and $\mathbf{x}_2$, relative to some reference, with components x_{1i} and x_{2i} respectively.

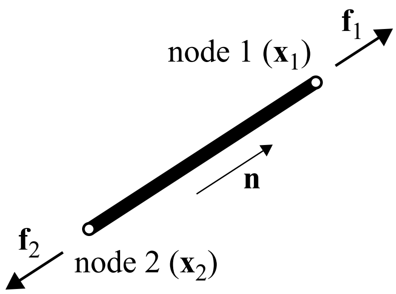

Figure 2: A single bar connecting two nodes, at positions $\mathbf{x}_1$ and $\mathbf{x}_2$; a unit vector along the bar from $\mathbf{x}_2$ to $\mathbf{x}_1$ is given by $\mathbf{n}$.

The bar is currently of length l, and a unit vector $\mathbf{n} = (\mathbf{x}_1 - \mathbf{x}_2)/l$ is parallel to the bar.

Equilibrium at nodes 1 and 2 can be written in terms of the bar tension, t, in either vector, or component form,

$$\mathbf{f}_1 = \mathbf{n}\, t \quad ; \quad f_{1i} = n_i\, t \tag{1}$$

$$\mathbf{f}_2 = -\mathbf{n}\, t \quad ; \quad f_{2i} = -n_i\, t. \tag{2}$$

Alternatively, the equilibrium equations can be written using the *tension coefficient* in the bar, $\hat{t} = t/l$.

$$\mathbf{f}_1 = (\mathbf{x}_1 - \mathbf{x}_2)\hat{t} \quad ; \quad f_{1i} = (x_{1i} - x_{2i})\hat{t} \tag{3}$$

$$\mathbf{f}_2 = (-\mathbf{x}_1 + \mathbf{x}_2)\hat{t} \quad ; \quad f_{2i} = (-x_{1i} + x_{2i})\hat{t}. \tag{4}$$

In order to find the tangent stiffness, differentiating the component equilibrium expressions in (3) and (4) with respect to the j-coordinate of node 1 gives

$$\frac{\partial f_{1i}}{\partial x_{1j}} = (x_{1i} - x_{2i})\,\frac{\partial \hat{t}}{\partial x_{1j}} + \delta_{ij}\,\hat{t} \tag{5}$$

$$\frac{\partial f_{2i}}{\partial x_{1j}} = (-x_{1i} + x_{2i})\,\frac{\partial \hat{t}}{\partial x_{1j}} - \delta_{ij}\,\hat{t} \tag{6}$$

where $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$. Similarly differentiating with respect to the j-coordinate of node 2 gives

$$\frac{\partial f_{1i}}{\partial x_{2j}} = (x_{1i} - x_{2i})\,\frac{\partial \hat{t}}{\partial x_{2j}} - \delta_{ij}\,\hat{t} \tag{7}$$

$$\frac{\partial f_{2i}}{\partial x_{2j}} = (-x_{1i} + x_{2i})\,\frac{\partial \hat{t}}{\partial x_{2j}} + \delta_{ij}\,\hat{t}. \tag{8}$$

To simplify the stiffness expressions (5)–(8) requires further expansion of the rate of change of the tension coefficient with position of the nodes. A basic assumption for pin-jointed bars if that the tension in a particular bar varies only with the extension, or equivalently the length, of that bar. It is thus sensible to write

$$\frac{\partial \hat{t}}{\partial x_{1j}} = \frac{d\hat{t}}{dl}\frac{\partial l}{\partial x_{1j}} \quad ; \quad \frac{\partial \hat{t}}{\partial x_{2j}} = \frac{d\hat{t}}{dl}\frac{\partial l}{\partial x_{2j}} \tag{9}$$

where trigonometry shows that

$$\frac{\partial l}{\partial x_{1j}} = n_j \quad ; \quad \frac{\partial l}{\partial x_{2j}} = -n_j, \tag{10}$$

and hence

$$\frac{\partial \hat{t}}{\partial x_{1j}} = \frac{d\hat{t}}{dl} n_j \quad ; \quad \frac{\partial \hat{t}}{\partial x_{2j}} = -\frac{d\hat{t}}{dl} n_j. \tag{11}$$

The rate of change of tension coefficient with length, $d\hat{t}/dl$, can be written as

$$\begin{aligned}\frac{d\hat{t}}{dl} = \frac{d(t/l)}{dl} &= \frac{1}{l}\frac{dt}{dl} - \frac{t}{l^2} \\ &= \frac{1}{l}\left(\frac{dt}{dl} - \hat{t}\right). \end{aligned} \tag{12}$$

The rate of change of tension with respect to length of the bar, dt/dl, is simply the axial stiffness. For small strains of a linear-elastic bar with cross-sectional area A, Young's modulus E, and initial length l_0, $dt/dl = AE/l_0$. However, we will only assume that the tension is differentiable, although this does imply that we are dealing with an elastic system, and don't have a cable at its rest-length. Within this assumption, to maintain generality, we will define the axial stiffness, dt/dl, as g, a bar parameter that may vary with bar length, giving

$$\frac{d\hat{t}}{dl} = \frac{g - \hat{t}}{l} \tag{13}$$

To simplify notation further, define a *modified axial stiffness*, $\hat{g} = g - \hat{t}$, giving

$$\frac{d\hat{t}}{dl} = \frac{\hat{g}}{l} \tag{14}$$

Substituting (14) into (11) gives

$$\frac{\partial \hat{t}}{\partial x_{1j}} = \frac{\hat{g} n_j}{l} \quad ; \quad \frac{\partial \hat{t}}{\partial x_{2j}} = -\frac{\hat{g} n_j}{l} \tag{15}$$

and hence the stiffness equations (5)–(8) can be written, noting that $(x_{1i} - x_{2i})/l = n_i$, as

$$\frac{\partial f_{1i}}{\partial x_{1j}} = n_i\, \hat{g}\, n_j + \delta_{ij}\, \hat{t} \quad ; \quad \frac{\partial f_{1i}}{\partial x_{2j}} = -n_i\, \hat{g}\, n_j - \delta_{ij}\, \hat{t} \tag{16}$$

$$\frac{\partial f_{2i}}{\partial x_{1j}} = -n_i\, \hat{g}\, n_j - \delta_{ij}\, \hat{t} \quad ; \quad \frac{\partial f_{2i}}{\partial x_{2j}} = n_i\, \hat{g}\, n_j + \delta_{ij}\, \hat{t} \tag{17}$$

or, in vector form

$$\frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_1} = \mathbf{n}\, \hat{g}\, \mathbf{n}^{\mathrm{T}} + \hat{t}\, \mathbf{I} \quad ; \quad \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_2} = -\mathbf{n}\, \hat{g}\, \mathbf{n}^{\mathrm{T}} - \hat{t}\, \mathbf{I} \tag{18}$$

$$\frac{\partial \mathbf{f}_2}{\partial \mathbf{x}_1} = -\mathbf{n}\, \hat{g}\, \mathbf{n}^{\mathrm{T}} - \hat{t}\, \mathbf{I} \quad ; \quad \frac{\partial \mathbf{f}_2}{\partial \mathbf{x}_2} = \mathbf{n}\, \hat{g}\, \mathbf{n}^{\mathrm{T}} + \hat{t}\, \mathbf{I}. \tag{19}$$

Thus, for a single bar, the tangent stiffness matrix, $\mathbf{K}_s$, relating small changes in nodal position to small changes in nodal forces,

$$\begin{bmatrix} \delta \mathbf{f}_1 \\ \delta \mathbf{f}_2 \end{bmatrix} = \mathbf{K}_s \begin{bmatrix} \delta \mathbf{x}_1 \\ \delta \mathbf{x}_2 \end{bmatrix} \tag{20}$$

is given by

$$\mathbf{K}_s = \begin{bmatrix} \mathbf{n} \\ -\mathbf{n} \end{bmatrix} \begin{bmatrix} \hat{g} \end{bmatrix} \begin{bmatrix} \mathbf{n}^{\mathrm{T}} & -\mathbf{n}^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} \hat{t}\mathbf{I} & -\hat{t}\mathbf{I} \\ -\hat{t}\mathbf{I} & \hat{t}\mathbf{I} \end{bmatrix} \tag{21}$$

which can be written as

$$\mathbf{K}_s = \mathbf{a}_s \begin{bmatrix} \hat{g} \end{bmatrix} \mathbf{a}_s^{\mathrm{T}} + \mathbf{S}_s \tag{22}$$

where $\mathbf{a}_s$ is the *equilibrium matrix* for a single bar,

$$\mathbf{a}_s = \begin{bmatrix} \mathbf{n} \\ -\mathbf{n} \end{bmatrix} \tag{23}$$

relating bar tension and nodal force,

$$\mathbf{a}_s \begin{bmatrix} t \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} \tag{24}$$

and $\mathbf{S}_s$ is the *stress matrix* for a single bar,

$$\mathbf{S}_s = \begin{bmatrix} \hat{t}\mathbf{I} & -\hat{t}\mathbf{I} \\ -\hat{t}\mathbf{I} & \hat{t}\mathbf{I} \end{bmatrix}. \tag{25}$$

2.2 Complete structure

We can find the tangent stiffness matrix for an entire structure simply by adding together the tangent stiffness matrices for individual bars. To do this the tangent stiffness matrices for individual bars must first be embedded in a larger coordinate system for the entire structure.

Consider a structure consisting of n nodes. Define a vector of nodal forces $\mathbf{f}$ and a vector of nodal coordinates $\mathbf{x}$, where

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_n \end{bmatrix} \quad ; \quad \mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{bmatrix} \tag{26}$$

and $\mathbf{f}_i$ is the two- or three-dimensional force vector at node i, and $\mathbf{x}_i$ is the two- or three-dimensional position vector of node i.

Consider a bar p connecting nodes i and j with current length l_p, carrying a tension t_p, a tension coefficient $\hat{t}_p$, and having a modified axial stiffness $\hat{g}_p$. Define a unit vector $\mathbf{n}_{ij}$ along bar p,

$$\mathbf{n}_{ij} = \frac{\mathbf{x}_i - \mathbf{x}_j}{l_p} = -\mathbf{n}_{ji} \tag{27}$$

The equilibrium matrix for this bar in the global coordinate system, $\mathbf{a}_p$, is defined so that the nodal forces $\mathbf{f}_p$ in equilibrium with a tension t_p in bar p are given by

$$\mathbf{a}_p \begin{bmatrix} t_p \end{bmatrix} = \mathbf{f}_p, \tag{28}$$

and hence has all components zero, apart from those corresponding to the nodes at the end of the bar, i and j,

$$\mathbf{a}_p = \begin{bmatrix} \mathbf{a}_{p1} \\ \mathbf{a}_{p2} \\ \vdots \\ \mathbf{a}_{pn} \end{bmatrix} \quad ; \quad \mathbf{a}_{pi} = \mathbf{n}_{ij} \quad ; \quad \mathbf{a}_{pj} = \mathbf{n}_{ji} = -\mathbf{n}_{ij} \quad ; \quad \mathbf{a}_{pk} = \mathbf{0} \text{ if } k \neq i \text{ and } k \neq j. \tag{29}$$

The stress matrix for the single bar p joining nodes i and j, in a global coordinate system, can be defined in terms of 2 by 2 (in 2D) or 3 by 3 (in 3D) submatrices $\mathbf{s}_{p_{lm}}$,

$$\mathbf{S}_p = \begin{bmatrix} \mathbf{s}_{p_{11}} & \mathbf{s}_{p_{12}} & \cdots & \mathbf{s}_{p_{1n}} \\ \mathbf{s}_{p_{21}} & \mathbf{s}_{p_{22}} & & \\ \vdots & & \ddots & \\ \mathbf{s}_{p_{n1}} & & & \mathbf{s}_{p_{nn}} \end{bmatrix} \tag{30}$$

where

$$\mathbf{s}_{p_{ii}} = \mathbf{s}_{p_{jj}} = \hat{t}_p \mathbf{I} \quad ; \quad \mathbf{s}_{p_{ij}} = \mathbf{s}_{p_{ji}} = -\hat{t}_p \mathbf{I}, \tag{31}$$

and all other $\mathbf{s}_{p_{lm}} = \mathbf{0}$.

The tangent stiffness matrix for bar p, $\mathbf{K}_p$, can be written by embedding (22) in the global coordinate system,

$$\mathbf{K}_p = \mathbf{a}_p \left[\hat{g}_p\right] \mathbf{a}_p^{\mathrm{T}} + \mathbf{S}_p. \tag{32}$$

Consider a structure made up of b bars. The total tangent stiffness, $\mathbf{K}$, can be found by adding up the tangent stiffness due to each of the bars

$$\mathbf{K} = \sum_{p=1}^{b} \mathbf{K}_p = \sum_{p=1}^{b} \mathbf{a}_p \left[\hat{g}_p\right] \mathbf{a}_p^{\mathrm{T}} + \sum_{p=1}^{b} \mathbf{S}_p \tag{33}$$

which can be written as

$$\mathbf{K} = \mathbf{A}\hat{\mathbf{G}}\mathbf{A}^{\mathrm{T}} + \mathbf{S} \tag{34}$$

where $\mathbf{A}$ is the *equilibrium matrix* for the entire structure

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_b \end{bmatrix}, \tag{35}$$

$\hat{\mathbf{G}}$ is a diagonal matrix of modified axial stiffnesses,

$$\hat{\mathbf{G}} = \begin{bmatrix} \hat{g}_1 & & & \\ & \hat{g}_2 & & \\ & & \ddots & \\ & & & \hat{g}_b \end{bmatrix}, \tag{36}$$

and $\mathbf{S}$ is the *stress matrix* for the entire structure. $\mathbf{S}$ can be defined in terms of component (2×2) or (3×3) submatrices $\mathbf{s}_{lm}$

$$\mathbf{S} = \begin{bmatrix} \mathbf{s}_{11} & \mathbf{s}_{12} & \cdots & \mathbf{s}_{1n} \\ \mathbf{s}_{21} & \mathbf{s}_{22} & & \\ \vdots & & \ddots & \\ \mathbf{s}_{n1} & & & \mathbf{s}_{nn} \end{bmatrix} \tag{37}$$

where, for $l = m$,

$$\mathbf{s}_{ll} = \hat{t}_{ll}\mathbf{I}, \tag{38}$$

and $\hat{t}_{ll}$ is the sum of the tension coefficients of all the bars that meet at node l, and, for $l \neq m$,

$$\mathbf{s}_{lm} = -\hat{t}_{lm}\mathbf{I}, \tag{39}$$

where $\hat{t}_{lm}$ is equal to the tension coefficient in the bar joining node i and j if the nodes are connected by a bar, or is zero otherwise.

3 Comparison with other formulations

3.1 Conventional stiffness/geometric stiffness formulation

A conventional formulation of the tangent stiffness would describe the tangent stiffness as consisting of two parts, a material stiffness, and a geometric stiffness. The material stiffness corresponds to the stiffness when it is assumed that the overall geometry of the structure does not change due to load, or alternatively that the structure is initially unstressed. The geometric stiffness corresponds to the stiffness due to the reorientation of stressed members. It is instructive to compare these terms with the new formulation. This can easily be done by differentiating equilibrium expressions, as in Section 2. However, in contrast to the new derivation in Section 2, we will work directly with tension in the bar as a variable, rather than forming the tension coefficient.

Starting with the equilibrium expressions, (1) and (2), differentiating with respect to the position of the nodes gives

$$\frac{\partial f_{1i}}{\partial x_{1j}} = n_i \frac{\partial t}{\partial x_{1j}} + \frac{\partial n_i}{\partial x_{1j}} t \quad ; \quad \frac{\partial f_{1i}}{\partial x_{2j}} = n_i \frac{\partial t}{\partial x_{2j}} + \frac{\partial n_i}{\partial x_{2j}} t \tag{40}$$

$$\frac{\partial f_{2i}}{\partial x_{1j}} = -n_i \frac{\partial t}{\partial x_{1j}} - \frac{\partial n_i}{\partial x_{1j}} t \quad ; \quad \frac{\partial f_{2i}}{\partial x_{2j}} = -n_i \frac{\partial t}{\partial x_{2j}} - \frac{\partial n_i}{\partial x_{2j}} t \tag{41}$$

The first term in each of the expressions, e.g. $(n_i\, \partial t/\partial x_{1j})$, together make up the material stiffness matrix. Following similar working to that in Section 2, it is possible to finally write, for a single bar, the material stiffness matrix $\mathbf{K}_{sm}$ in the form,

$$\mathbf{K}_{sm} = \mathbf{a}_s \left[g\right] \mathbf{a}_s^{\mathrm{T}}, \tag{42}$$

and for the complete structure the material stiffness matrix, $\mathbf{K}_m$, is given by,

$$\mathbf{K}_m = \mathbf{A}\mathbf{G}\mathbf{A}^{\mathrm{T}}. \tag{43}$$

The geometric stiffness matrix, $\mathbf{K}_g$, can be derived from the difference between (43) and (34),

$$\mathbf{K}_g = -\mathbf{A}\hat{\mathbf{T}}\mathbf{A}^{\mathrm{T}} + \mathbf{S}, \tag{44}$$

where $\hat{\mathbf{T}}$ is the diagonal matrix of tension coefficients. Thus, part of the geometric stiffness has exactly the same structure as the material stiffness matrix; the new formulation (34) lumps these terms together.

For most conventional structures, it is reasonable to assume that the modified axial stiffness for any bar $\hat{g}$, will be little different to the axial stiffness g. As a 'worst case', consider a linear-elastic bar with axial stiffness AE/l that carries tension just less than that required to cause yield. The tension will be given by $AE\epsilon_y$, where ϵ_y is the yield strain, and thus the modified axial stiffness is $\hat{g} = g - t/l = (AE/l)(1 - \epsilon_y)$. Thus, for bars where $\epsilon_y << 1$, the modified axial stiffness will be little different from the conventional axial stiffness, and certainly positive. This is not universally true, however. For instance it is possible for wound springs to have zero modified axial stiffness, by ensuring that in an initial, closely wound, state they carry a tension equivalent to having a zero rest length, a principle used to advantage in Anglepoise lamps (French and Widden, 2000).

3.2 Equilibrium matrices and the product-force approach

Equation (43) can be considered as the decomposition of the material stiffness matrix into compatibility, equilibrium, and bar-stiffness relationships. The equilibrium matrix and the bar stiffness relationships have already been described. It is straightforward to show, be e.g. a virtual work argument, that the transpose of the equilibrium matrix, $\mathbf{A}^{\mathrm{T}}$, is the *compatibility matrix* for the structure, also known as the *rigidity matrix*, relating extensions of the bars to displacement of nodes. Consider a vector, $\mathbf{e}$, of bar extensions, relative to the current configuration,

$$\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_b \end{bmatrix} = \begin{bmatrix} \delta l_1 \\ \delta l_2 \\ \vdots \\ \delta l_b \end{bmatrix} \tag{45}$$

and a vector $\mathbf{d}$ of nodal displacements

$$\mathbf{d} = \delta \mathbf{x} = \begin{bmatrix} \delta \mathbf{x}_1 \\ \delta \mathbf{x}_2 \\ \vdots \\ \delta \mathbf{x}_n \end{bmatrix}. \tag{46}$$

$\mathbf{e}$ and $\mathbf{d}$ are related by

$$\mathbf{e} = \mathbf{A}^{\mathrm{T}}\mathbf{d} \tag{47}$$

The nullspace of $\mathbf{A}^{\mathrm{T}}$ contains all *mechanisms* of the structure, nodal displacements corresponding to zero member extension. If the nullspace is m-dimensional, the mechanisms can be described by a set of basis vectors, $\mathbf{m}_1 \ldots \mathbf{m}_m$. If these mechanisms are written as the columns of a matrix $\mathbf{D}$,

$$\mathbf{D} = \begin{bmatrix} \mathbf{m}_1 & \mathbf{m}_2 & \cdots & \mathbf{m}_m \end{bmatrix} \tag{48}$$

then a general mechanism $\mathbf{m}$ is given by

$$\mathbf{m} = \mathbf{D}\mathbf{b} \tag{49}$$

where $\mathbf{b}$ gives the coefficient of each of the basis mechanisms.

For any mechanism, the material stiffness matrix gives zero stiffness. The 'material' force developed as the mechanism is displaced is given by $\mathbf{K}_m\mathbf{m}$, where

$$\begin{aligned} \mathbf{K}_m\mathbf{m} &= \mathbf{A}\mathbf{G}\mathbf{A}^{\mathrm{T}}\mathbf{m} \\ &= \mathbf{A}\mathbf{G}\mathbf{A}^{\mathrm{T}}\mathbf{D}\mathbf{b} \\ &= \mathbf{0} \end{aligned} \tag{50}$$

as $\mathbf{A}^{\mathrm{T}}\mathbf{D} = \mathbf{0}$. This result is also true if the modified axial stiffness is used, $\mathbf{A}\hat{\mathbf{G}}\mathbf{A}^{\mathrm{T}}\mathbf{m} = \mathbf{0}$. However, this does not imply that the stiffness of a mechanism is zero, and Calladine and Pellegrino (1991) introduced a method to find this stiffness. The actual (linearized) force developed as any mechanism is actuated is given, using the complete tangent stiffness matrix (34), by

$$\begin{aligned} \mathbf{f} = \mathbf{K}\mathbf{m} = \mathbf{K}\mathbf{D}\mathbf{b} &= \mathbf{A}\hat{\mathbf{G}}\mathbf{A}^{\mathrm{T}}\mathbf{D}\mathbf{b} + \mathbf{S}\mathbf{D}\mathbf{b} \\ &= \mathbf{S}\mathbf{D}\mathbf{b}, \end{aligned} \tag{51}$$

and the (linearized) work done during the deformation is given by

$$\begin{aligned} W &= \frac{1}{2}\mathbf{d}^{\mathrm{T}}\mathbf{f} \\ &= \frac{1}{2}\mathbf{b}^{\mathrm{T}}\mathbf{D}^{\mathrm{T}}\mathbf{K}\mathbf{D}\mathbf{b} \\ &= \frac{1}{2}\mathbf{b}^{\mathrm{T}}\mathbf{D}^{\mathrm{T}}\mathbf{S}\mathbf{D}\mathbf{b}. \end{aligned} \tag{52}$$

Calladine and Pellegrino (1991) described the matrix $\mathbf{D}^{\mathrm{T}}\mathbf{S}\mathbf{D}$ as a matrix $\mathbf{Q}$; they commented that it was symmetric, which it clearly is from this derivation. It is natural to consider $\mathbf{Q}$ as a *reduced form* of the stress matrix, where motion is restricted only to to inextensional mechanisms of the structure. If $\mathbf{Q}$ is positive definite, there is positive stiffness for any mechanism of the structure.

3.3 Rigidity Theory and Prestress Stability

Connelly and Whiteley (1996) clearly anticipate the results in this paper by showing that a full account of structural stiffness comes from two sources, a first order rigidity that can

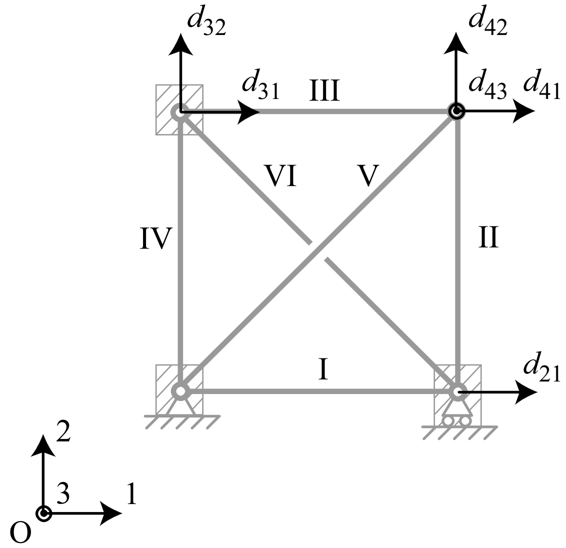

Figure 3: Coordinate systems for the simple example structure. The bars are numbered I–VI, and any allowed displacement of node i in direction j is denoted d_{ij}.

be written in terms of the rigidity matrix (the transpose of the equilibrium matrix), and a term given by the stress matrix. However, the link with tangent stiffness formulations if not immediately clear, largely because this work is not concerned with finding particular numerical values, but rather with answering general questions about structural stability. A further problem arises because of differences in notation, particularly as the rigidity theory literature uses the term ‘stress’ for what is defined in this paper as a ‘tension coefficient’.

Equation (34) can be considered as a translation of the stiffness formulation given by Connelly and Whiteley (1996) into more conventional engineering terms. The key point is that the basic structure of the equations is the same, and this means that many of the further powerful results in Connelly and Whiteley (1996), and related literature, can be directly translated and understood in conventional engineering terms.

4 Example

This section will analyze in three dimensions the structure shown in Figure 1, using the new formulation of the tangent stiffness matrix. Figure 3 shows the same structure, but with a coordinate system added for possible nodal displacements, along with a bar numbering scheme. We define forces $f_{21} \ldots f_{43}$ to be forces work equivalent to the displacements $d_{21} \ldots d_{43}$ shown.

The equilibrium matrix for the structure relates external forces to bar tensions, $\mathbf{At} = \mathbf{f}$, and is given by,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & -1 & 0 & 0 & -1/\sqrt{2} \\ 0 & 0 & 0 & 1 & 0 & 1/\sqrt{2} \\ 0 & 0 & 1 & 0 & 1/\sqrt{2} & 0 \\ 0 & 1 & 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t_{\mathrm{I}} \\ t_{\mathrm{II}} \\ t_{\mathrm{III}} \\ t_{\mathrm{IV}} \\ t_{\mathrm{V}} \\ t_{\mathrm{VI}} \end{bmatrix} = \begin{bmatrix} d_{21} \\ d_{31} \\ d_{32} \\ d_{31} \\ d_{32} \\ d_{33} \end{bmatrix} \tag{53}$$

Although the matrix is square, it is clearly rank-deficient, and the null-space gives the state of self-stress in the system, $\mathbf{t}_0$, in terms of an arbitrary constant, the tension T in bar I,

$$\mathbf{t}_0 = \begin{bmatrix} T \\ T \\ T \\ T \\ -\sqrt{2}T \\ -\sqrt{2}T \end{bmatrix} \tag{54}$$

and hence, when the structure is unloaded, the tension coefficients in the bars are given by

$$\begin{bmatrix} t_{\mathrm{I}} \\ t_{\mathrm{II}} \\ t_{\mathrm{III}} \\ t_{\mathrm{IV}} \\ t_{\mathrm{V}} \\ t_{\mathrm{VI}} \end{bmatrix} = \frac{T}{L} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}. \tag{55}$$

Thus the modified axial stiffness matrix is given by

$$\hat{\mathbf{G}} = \frac{AE}{L} \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1/\sqrt{2} & \\ & & & & & 1/\sqrt{2} \end{bmatrix} - \frac{T}{L} \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & -1 \end{bmatrix} \tag{56}$$

The structure of the stress matrix is most clearly seen by first considering the stress matrix of an identical structure that has been freed from its foundations, $\mathbf{S}_f$. At each node, the sum of the tension coefficients in the bars, $\hat{t}_{ii} = (1+1-1)T/L$, which forms the diagonal terms in $\mathbf{S}_f$. The off-diagonal terms $\hat{t}_{ij}$ are the negative of the tension coefficient in the bar, and are hence $+T/L$ for the diagonal bars, and $-T/L$ for the others, giving

$$\mathbf{S}_f = \frac{T}{L} \begin{bmatrix} \mathbf{I} & -\mathbf{I} & \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{I} & -\mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} & \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{I} & -\mathbf{I} & \mathbf{I} \end{bmatrix} \tag{57}$$

where $\mathbf{I}$ is a 3×3 identity matrix. The stress matrix for the actual restrained case can be found by crossing out the rows and columns corresponding to restrained degrees of freedom, leaving

$$\mathbf{S} = \frac{T}{L} \left[\begin{array}{c:cc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ \hdashline -1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ \hdashline 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]. \tag{58}$$

Substituting the equilibrium matrix from (53), the matrix of modified axial stiffness given in (56) and the stress matrix given in (58) into the complete tangent stiffness formulation (34) gives the complete tangent stiffness matrix for the structure, $\mathbf{K} = \mathbf{A}\hat{\mathbf{G}}\mathbf{A}^{\mathrm{T}} + \mathbf{S}$.

The nullspace of the transposed equilibrium matrix for the structure describes the one mechanism,

$$\mathbf{m} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{59}$$

And clearly for this mechanism, $\mathbf{A}\hat{\mathbf{G}}\mathbf{A}^{\mathrm{T}}\mathbf{m} = \mathbf{0}$ (as $\mathbf{A}^{\mathrm{T}}\mathbf{m} = \mathbf{0}$). Thus any stiffness must be given by the stress matrix term, which gives the force (the product-force) as

$$\mathbf{f} = \mathbf{S}\mathbf{m} = \frac{T}{L} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{60}$$

and indeed, the reduced (1×1) stress matrix corresponding to this mechanism is given by

$$\mathbf{S}_r = \mathbf{m}^{\mathrm{T}}\mathbf{S}\mathbf{m} = \frac{T}{L} \tag{61}$$

Thus the structure will have positive stiffness in all modes as long as T is positive, i.e. the outer four bars are in tension, while the inner two bars are in compression.

5 Discussion

The tangent stiffness formulation presented in this paper is certainly not new. It first appeared in Argyris and Scharpf (1972), and has been used in much work since; a recent equivalent but extended derivation in a large displacement, large strain, setting has been given by Murakami (2001a), using the powerful tools of continuum mechanics. Although the final formulation is not new, the present paper does give a new and simple derivation of the tangent stiffness, and writes it in a form which allows comparison with other formulations in the literature. A novel feature is the use of a modified axial stiffness, which for conventional structures is little different from the conventional axial stiffness.

An important feature of this paper is that it links into the work in mathematical rigidity theory. This line of research is often neglected in the engineering literature, despite the powerful results that have been derived. This may partly be because of difficulties of notation, as well as the different underlying aims of the work. This paper has shown that in fact the stiffness formulation given e.g. by Connelly and Whiteley (1996) is directly compatible with a standard tangent stiffness formulation.

The paper also shows the links between tangent stiffness of a prestressed structure, and the product force method of Pellegrino and Calladine, a link that has also been made by Murakami (2001b). Recently, Tarnai and Szabó (2002) have elucidated the link between the product force method and the geometric formulations of Kuznetsov, (e.g. Kuznetsov, 1991): together with the results in this paper this gives further unification to seemingly disparate methods.

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