One and Half Hours
Reading time  10 Minutes

Answer all questions.

The approximate number of marks for each section of a question is indicated in the right hand margin.
Start each question on a separate sheet and write your name at the top of each sheet.
Departmental Data Books and standard calculators are to be used.
SECTION A

1 (short) Express \[ \frac{\sin x}{\sqrt{1 + 2x^2}} \] as a power series in \( x \), up to and including the term in \( x^3 \). [10]

2 (short) Find the general solution of the differential equation:
\[ \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} - 4y = 5e^x. \] [10]

3 (long)

(a) Find the eigenvalues and eigenvectors of the matrix \( A \), where
\[ A = \frac{1}{4} \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
Hence describe the geometrical transformation represented by \( A \). [12]

(b) If \( x = [1 \ 2 \ 1]^T \), what, approximately, is \( A^{20}x \)? [6]

(c) The matrix \( B \) has the same eigenvectors as \( A \) but distinct eigenvalues \( \lambda_{B1}, \lambda_{B2} \) and \( \lambda_{B3} \), which are each different from those of \( A \).

(i) Prove that \( AB = BA \). [6]

(ii) What are the eigenvalues and eigenvectors of \( AB \)? [6]
4 (short) 30% of the population carry the MRSA bacterium. A diagnostic test detects MRSA in 95% of carriers and (incorrectly) 2% of non-carriers. I have just tested positive for MRSA: what is the probability that I am a carrier? [10]

5 (short) A linear system has step response $1 - e^{-t}$ for $t \geq 0$. Find its response to the input

$$x(t) = \begin{cases} 0 & t < T \\ 2 - 3\delta(t - 2T) & t \geq T \end{cases}$$

where $T > 0$. Is this a first or second order system? Justify your answer. [10]

6 (long) Consider the function

$$f(t) = \begin{cases} \pi & -\pi \leq t < 0 \\ \pi - t & 0 \leq t < \pi \end{cases}$$

where $f(t)$ is periodic with period $2\pi$.

(a) Sketch $f(t)$ and $f'(t)$ in the range $-2\pi \leq t \leq 2\pi$. Take care to mark the locations and magnitudes of any $\delta$ functions on the sketch of $f'(t)$. [6]

(b) What is the mean value of $f'(t)$? [4]

(c) Show that the Fourier series for $f'(t)$ is

$$f'(t) = \sum_{n=1}^{\infty} \left( (-1)^n \cos nt + \left( \frac{(-1)^n - 1}{n\pi} \right) \sin nt \right)$$

[10]

(d) Find a Fourier series for $f(t)$. [6]

(e) Verify that the series for $f'(t)$ and $f(t)$ converge, or otherwise, at the expected rates. [4]

End of Paper